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CONTINUOUS CATEGORIES AND EXPONENTIABLE TOPOSES

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Introduction

The main objective of this paper is to contribute to the study of toposes as 'generalized spaces', by obtaining conditions for the existence of 'function spaces' (i.e. exponentials) in the 2-category of (Grothendieck) toposes and geometric morphisms. In view of the importance of function spaces in many areas of topology, we feel that this objective requires little justification. However, in analysing the notion of function space in the topos-theoretic context, we have been led to introduce a new concept which we have christened a 'continuous category', and which seems likely to be of considerable independent interest for the light which it sheds on the rapidly growing subject of continuous lattices. Accordingly, the first two sections of the paper are devoted to developing this new concept, and it therefore seems worthwhile to preface them with a reasonably non-technical account of how it has arisen.

It has been known for many years [6] that, if X and Y are spaces, the space Y^X of continuous functions from X to Y has its most pleasing properties if X is locally compact. It was first pointed out by Day and Kelly [2] that this good behaviour is related to a certain lattice-theoretic property of the open-set lattices of locally compact spaces. Subsequently, lattices with this property were studied (and given the name 'continuous lattices') by Scott [33], for rather different reasons: his researches in the theory of computation led him to regard them as a natural generalization of algebraic lattices (in fact the completion of the latter under splitting of idempotents in a suitable category of 'continuous maps'). More strikingly, Scott also showed that every continuous lattice admits a certain intrinsic topology, and that the spaces obtained in this way from continuous lattices are

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precisely the injective objects (with respect to subspace inclusions) in the category of T_0-spaces and continuous maps.

Later, work of Isbell [19], Hofmann and Lawson [15] and Banaschewski [1] emphasized the links between local compactness, exponentiability (=possessing well-behaved function-spaces), and having continuous open-set lattice. Hyland [17] took a further step in this direction when he replaced the category of spaces by that of locales (i.e. complete Heyting algebras, regarded as generalized open-set lattices [18]); he was able to show that the link between continuous lattices and exponentiability remained valid in this context. However, none of these authors made explicit the link between exponentiability and injectivity in the category of T_0-spaces (or of locales); since this link is one of the important elements in our approach to exponentiability, we sketch it here.

Suppose X is a T_0-space (or a locale, according to taste) which is exponentiable; i.e., the functor $(-)\times X$ has a right adjoint $(-)^X$. Let S denote the Sierpiński space, i.e. the two-point space with just one open point. It is trivial to verify that S is injective (see [33]). Moreover, the functor $(-)\times X$ preserves subspace inclusions, so its right adjoint $(-)^X$ must preserve injectives; hence S^X is injective, and by Scott's theorem its points form a continuous lattice in a canonical way. But by the adjunction, points of S^X correspond bijectively to continuous maps $X\to S$, and hence to open subsets of X; so these too form a continuous lattice. (Of course, it is necessary to check that the canonical ordering on points of S^X coincides with the inclusion ordering on open subsets of X; we omit the details.)

In the above argument, we used only the existence of the particular exponential S^X. But this is no accident: since S is a cogenerator for the category of sober spaces, the existence of S^X implies the existence of Y^X for any sober space Y. In the converse direction, suppose we know that the open sets of X form a continuous lattice. Endowing this lattice with Scott's topology, we obtain an injective space which is clearly the only possible candidate for the exponential S^X. Once again, some further work is needed to show that this space does have the universal property of an exponential, and we shall not give the details here.

Let us now compare these arguments with what happens in the category $\mathfrak{BTop}/\mathcal{S}$ of Grothendieck toposes. (Here $\mathcal{S}$ denotes 'the' classical topos of sets, but in practice it could easily be replaced by any base topos having a natural number object.) The first thing to note is that, since $\mathfrak{BTop}/\mathcal{S}$ is a 2-category, we must concern ourselves with exponentiability in the 2-categorical sense; that is, we shall say a topos $\mathcal{E}$ is exponentiable if there exists a functor $(-)^{\mathcal{E}}$ and a natural equivalence of categories

$$\mathfrak{BTop}/\mathcal{S}(\mathcal{G}\times_{\mathcal{S}}\mathcal{E},\mathcal{F})\simeq\mathfrak{BTop}/\mathcal{S}(\mathcal{G},\mathcal{F}^{\mathcal{E}})$$

for any pair of toposes $(\mathcal{F},\mathcal{G})$ (rather than a bijection between the objects of these categories).

In $\mathfrak{BTop}/\mathcal{S}$ the role of Sierpiński space is played by the *object classifier* $\mathcal{S}[X]$ (see [24]); it is easily deduced from the universal property of this topos that it is injective (with respect to subtopos inclusions) and a cogenerator in a suitable 2-categorical sense. Accordingly, we may reduce the question of whether a topos $\mathscr{E}$ is exponentiable to that of whether the particular exponential $\mathscr{S}[X]^{\mathscr{E}}$ exists; and if this exponential exists it is necessarily an injective topos. But by the adjunction, the category of points of $\mathscr{S}[X]^{\mathscr{E}}$ (i.e. geometric morphisms $\mathscr{S} \to \mathscr{S}[X]^{\mathscr{E}}$) is equivalent to $\mathscr{E}$ itself. Conversely, if $\mathscr{E}$ is equivalent to the category of points of an injective topos, then that topos (which, as we shall see, is determined up to equivalence by its category of points) is the natural candidate for the exponential $\mathscr{S}[X]^{\mathscr{E}}$, and we shall show that it does indeed have the right universal property. Our main result on exponentiability may thus be summarized as follows:

Theorem. *For a bounded $\mathscr{S}$-topos $\mathscr{E}$, the following are equivalent:*

(i) *$\mathscr{E}$ is exponentiable in $\mathfrak{BTop}/\mathscr{S}$.*
(ii) *The exponential $\mathscr{S}[X]^{\mathscr{E}}$ exists.*
(iii) *$\mathscr{E}$ is equivalent to the category of points of an injective topos.*

The proof of this theorem will occupy Section 4 of this paper. However, before we embark on its proof, it is clearly advisable to study injective toposes and their categories of points in some detail. The first investigation of injective toposes was carried out by Johnstone [21]; although this investigation was incomplete in certain important respects, it did establish the fact that the injective toposes are precisely the retracts in $\mathfrak{BTop}/\mathscr{S}$ of functor categories $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ where $\mathscr{C}$ has finite limits. (Note: we shall refrain from using the usual exponential notation for functor categories, since we wish to reserve it for topos exponentials.) Thus the categories of points of injective toposes are retracts, in an appropriate category, of the categories of points of such functor categories – but it is well known that the category of points of $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ is equivalent to the category of flat (=left exact) covariant functors $\mathscr{C} \to \mathscr{S}$. And the categories which arise in this way are exactly the *locally finitely presentable* categories of Gabriel and Ulmer [7].

Thus we are led to seek a categorical characterization of the idempotent-completion of locally finitely presentable categories – which is very reminiscent of Scott's characterization of continuous lattices as the idempotent-completion of algebraic lattices. When we have this characterization, it turns out that we can reverse the implication in the last paragraph: if $\mathscr{E}$ is a retract of a locally finitely presentable category, then there exists an injective topos, determined up to equivalence by $\mathscr{E}$, whose category of points is equivalent to $\mathscr{E}$.

But there is another direction in which we can generalize this result. It was first pointed out by Markowsky [28] that the concept of continuous lattice has a natural generalization to posets which are not necessarily lattices, and that the resulting 'continuous posets' have a number of useful applications. In the same way, it seems profitable to develop our 'continuous categories' in a context which does not require the existence of finite limits or colimits, and this is what we shall do in Section 2. We shall then be able to extend our results on injective toposes to the class of all toposes which occur as retracts of presheaf toposes; the latter appears as the natural analogue of the 'projective sober spaces' of Hoffmann [14], i.e. the spaces obtained by endowing continuous posets with the Scott topology.

What then is our definition of a continuous category? Clearly, we should seek it by taking the definition of a continuous poset and generalizing it to suit the context where the underlying structure is a category rather than a partial order. But we must exercise some care here. The usual definition of a continuous poset or lattice is phrased in terms of the properties of a certain auxiliary relation (the 'way-below relation') on the elements of the poset; whilst an analogue of the way-below relation (the concept of 'wavy arrow') certainly exists in a continuous category, it seems hard to give an intrinsic characterization of it, and it is therefore not convenient to use it in a definition. We therefore fall back on another characterization of continuous posets (first used, for continuous lattices, by Hofmann and Stralka [16]), which is couched in terms of the existence of a certain adjoint functor, and which thus admits a very straightforward generalization from posets to categories.

The precise definition will be found at the beginning of Section 2. We devote Section 1 to reviewing the theory of ind-completions of categories (in the sense of Grothendieck [10]), which is required for the definition; although this first section does not contain any new results, our presentation is perhaps rather different from anything in the existing literature. Section 2 then develops the theory of continuous categories (including the calculus of wavy arrows) up to the proofs of the basic theorems about retractions. In Section 3 we apply this theory to the study of injective toposes; our results here extend those in the first author's earlier paper [21], and we have borrowed a number of ideas from that source.

Section 4 contains the proof of our main theorem on exponentiability of toposes. As we indicated earlier, a large part of this proof can be developed without using the notion of continuous category, and could therefore be read before the sections which precede it; but it did not seem worthwhile to separate this material from the rest of the proof. Finally, Section 5 seeks to relate our results directly to those on exponentiability of spaces and locales. Somewhat surprisingly, not every locally compact space gives rise to an exponentiable topos of sheaves; but we give a characterization of those which do, and show that they include all locally compact Hausdorff spaces and all coherent (=spectral [13]) spaces. We have not tackled the problem of finding conditions on a general site to ensure that it generates an exponentiable topos, but it seems likely that the methods of Section 5 could be adapted to this end.

Throughout the first four sections, we have sought to motivate our definitions and arguments by emphasizing the way in which they generalize the corresponding things in the poset/lattice/locale case. Although this involves a certain amount of duplication of well-known results, we hope the reader will find it helpful in grasping the new concepts which we have to present.

Finally, we should mention that Susan Niefield [30] has independently considered the problem of exponentiability in $\mathfrak{BTop}/\mathscr{S}$, for an arbitrary base $\mathscr{S}$. Her methods are quite different from ours, and in fact her main results (which are concerned with exponentiability of subtoposes of $\mathcal{S}$) are almost disjoint from ours; but it seems likely that the combination of the two approaches may lead to further developments of interest.

1. Ind-completions

The characterization of continuous posets to which we referred in the Introduction is concerned with the relation between a poset P and its poset $\mathrm{Idl}(P)$ of ideals; in the absence of finite meets and joins, we define an *ideal* of P to be a subset $I \subseteq P$ which is (upwards) directed and downwards-closed, i.e. satisfies

$$(\exists i)(i \in I),$$

$$(i, j \in I) \Rightarrow (\exists k)(k \in I, i \leq k \text{ and } j \leq k),$$

$$(i \in I \text{ and } j \leq i) \Rightarrow (j \in I).$$

For any $p \in P$, the set $\downarrow(p) = \{x \in P \mid x \leq p\}$ is an ideal; this defines an embedding $\downarrow(-) : P \to \mathrm{Idl}(P)$. We can think of $\mathrm{Idl}(P)$ as the result of freely adjoining directed joins to P, without having regard to any directed joins which may already exist in P.

The analogue for categories of this construction is the notion of ind-completion, which was introduced by Grothendieck [10]. Although the basic facts about ind-completions are exposed in [12], we shall find it convenient to devote the present section to summarizing those results which we shall need, both in order to emphasize the analogy with the ideal-completion for posets, and in order to present them in a form which is suitable for reinterpreting in the context of categories indexed over a base topos [31, 32] – which will be useful when we come to consider exponentiability of toposes. For the present, however, we shall assume (at least for notational purposes) that our base category is 'the' topos of constant sets, which we shall denote by $\mathcal{S}$.

Let $\mathcal{E}$ be a locally small category, i.e. one with a Hom-functor taking values in $\mathcal{S}$. An *ind-object* in $\mathcal{E}$ is defined to be a small filtered diagram in $\mathcal{E}$, i.e. a functor $I \to \mathcal{E}$ where I is a small filtered category. (We shall frequently denote an ind-object by the indexed family $(X_i)_{i \in I}$ of its vertices, suppressing any explicit mention of the transition maps $X_i \to X_{i'}$ induced by morphisms $i \to i'$ in I.) We think of $(X_i)_{i \in I}$ as a 'formal colimit' of the diagram $I \to \mathcal{E}$ which we wish to adjoint to $\mathcal{E}$, in the same way that we think of an ideal in a poset as a 'formal directed join'.

To each object X of $\mathcal{E}$, we associate the *constant ind-object* $y(X)$, which is simply the functor $1 \to \mathcal{E}$ which picks out the object X. In defining morphisms of ind-objects, we are guided by three principles: (i) y should be a full embedding of $\mathcal{E}$ in $\text{Ind-}\mathcal{E}$; (ii) each ind-object $(X_i)_{i \in I}$ should be the actual colimit in $\text{Ind-}\mathcal{E}$ of the constant objects $y(X_i)$, $i \in I$; and (iii) the constant ind-objects should be *finitely presentable*, i.e. the functors $\mathrm{Hom}_{\text{Ind-}\mathcal{E}}(y(X), -)$ should preserve filtered colimits.

Given these, we necessarily have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{E}}((X_i)_{i\in I},(Y_j)_{j\in J}) &\cong \varprojlim_i \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{E}}(y(X_i),(Y_j)_{j\in J}) && \text{by (ii)}\\ &\cong \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{E}}(y(X_i),y(Y_j)) && \text{by (iii)}\\ &\cong \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathscr{E}}(X_i,Y_j) && \text{by (i),}\end{aligned}$$

and so we take the last expression as a definition of the first. More explicitly, a morphism $f:(X_i)_{i\in I}\to(Y_j)_{j\in J}$ is a family $(f_i)_{i\in I}$, where each f_i is an equivalence class of morphisms from X_i to some Y_j (two such morphisms $g:X_i\to Y_j$ and $g':X_i\to Y_{j'}$ being equivalent iff there exists a diagram $(j\to j''\leftarrow j')$ in J such that

$$\begin{array}{ccc} X_i & \xrightarrow{\;g\;} & Y_j\\ {\scriptstyle g'}\downarrow & & \downarrow\\ Y_{j'} & \longrightarrow & Y_{j''}\end{array}$$

commutes), the f_i being required to satisfy the compatibility condition that if $i\to i'$ is a morphism of I and $X_{i'}\to Y_j$ is a representative of $f_{i'}$, then the composite $X_i\to X_{i'}\to Y_j$ is a representative of f_i. In terms of this description, it is easy to define composition of morphisms of ind-objects, and to verify that Ind-$\mathscr{E}$ is a category and $y:\mathscr{E}\to$ Ind-$\mathscr{E}$ a full embedding. Also, since we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{E}}((X_i)_{i\in I},(Y_j)_{j\in J}) &= \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathscr{E}}(X_i,Y_j)\\ &\cong \varprojlim_i \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{E}}(y(X_i),(Y_j)_{j\in J}),\end{aligned}$$

it is clear that an ind-object $(X_i)_{i\in I}$ is indeed the filtered colimit in Ind-$\mathscr{E}$ of the constant objects $y(X_i)$. (We shall verify the third of the three principles used above in Proposition 1.5 below, after we have considered the nature of arbitrary filtered colimits in Ind-$\mathscr{E}$.)

It is interesting to note that some authors (e.g. [3]) define a morphism of ind- (or pro-) objects to be an equivalence class of indexed families rather than an indexed family of equivalence classes; that is, they define a notion of 'representative' for a morphism of ind-objects $f:(X_i)_{i\in J}\to(Y_j)_{j\in J}$ which amounts to the choice of a representative for each of the equivalence classes f_i. Of course, if we do not assume the axiom of choice (as we must not, if we wish our results to be re-interpretable over an arbitrary base topos), there is no reason to suppose that such representatives should exist.

Another approach to ind-completions (which is much exploited in [12]) is to regard Ind-$\mathscr{E}$ as embedded in the functor category $[\mathscr{E}^{\mathrm{op}},\mathscr{S}]$ via the functor

$$(X_i)_{i\in I}\mapsto \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{E}}(y(-),(X_i)_{i\in I}).$$

It is not hard to see that this functor is full and faithful, and so we might identify

Ind-$\mathscr{E}$ with its image in $[\mathscr{E}^{\mathrm{op}}, \mathscr{S}]$, which is clearly the full subcategory of functors which are expressible as small filtered colimits of representable functors. (Under this identification, the functor $y:\mathscr{E}\to$ Ind-$\mathscr{E}$ becomes identified with the Yoneda embedding $\mathscr{E}\to[\mathscr{E}^{\mathrm{op}}, \mathscr{S}]$.) However, for our purposes this viewpoint will not be so convenient; because of our desire to avoid invoking the axiom of choice, we shall wish to regard each ind-object of $\mathscr{E}$ as coming equipped with a *particular* representation as a small filtered colimit of objects of $\mathscr{E}$. (If $\mathscr{E}$ were a small category, there would be no problem about this; but in most of our applications $\mathscr{E}$ will not be small.)

To demonstrate that we do indeed have a generalization of the notion of ideal-completion, we begin by proving:

Lemma 1.1. *Let P be a (small) poset, regarded as a category. Then* Ind-P *is a preorder, and is equivalent as a category to* Idl(P).

Proof. The fact that Ind-P is a preorder follows easily from the 'double limit' definition of its hom-sets given earlier. If I is any ideal of P, then since I itself is directed we may regard the inclusion $I\to P$ as an ind-object of P; conversely if $\varphi:J\to P$ is any ind-object, then the downward-closure of the image of φ is an ideal of P. It is not hard to verify that these two constructions are functorial, and that they define an equivalence between Ind-P and Idl(P). □

Since our aim in constructing Ind-$\mathscr{E}$ was to adjoin filtered colimits to $\mathscr{E}$, we should certainly hope that Ind-$\mathscr{E}$ has filtered colimits. So our next task is to show that it does.

Theorem 1.2. *For any locally small category $\mathscr{E}$,* Ind-$\mathscr{E}$ *has (small) filtered colimits.*

Proof. Let $T:I\to$ Ind-$\mathscr{E}$ be a small filtered diagram in Ind-$\mathscr{E}$, and suppose each $T(i)=(X_{ij})_{j\in J_i}$. First we define a small category K as follows: its objects are pairs (i,j) with $j\in \mathrm{ob}\, J_i$, and morphisms $(i,j)\to(i',j')$ are pairs (α,f) where $\alpha:i\to i'$ in I and $f:X_{ij}\to X_{i'j'}$ is a representative for the jth component of $T(\alpha):T(i)\to T(i')$. Note in particular that for each $\beta:j\to j'$ in J_i, $X_{i\beta}:X_{ij}\to X_{ij'}$ is a representative for the jth component of the identity map on $T(i)$, so we have a functor $u_i:J_i\to K$ (not necessarily an embedding) which sends β to $(\mathrm{id}_i, X_{i\beta})$. (We can think of K as being something like a lax colimit [36] of the categories J_i, $i\in I$, except that the 'transition maps' induced by morphisms of I are not honest functors.)

We claim that K is a filtered category: we give the verification of the third condition for filteredness (the other two being similar). Let

$$(i,j)\underset{(\alpha',f')}{\overset{(\alpha,f)}{\rightrightarrows}}(i',j')$$

be a parallel pair of maps in K. Since I is filtered, we can find $\beta:i'\to i''$ in I with

$\beta\alpha = \beta\alpha'$; let $g : X_{i'j'} \to X_{i''j''}$ be any representative of the (j')th component of $T(\beta)$. Then the composites gf and gf' both represent the jth component of $T(\beta\alpha) = T(\beta\alpha')$, so we can find $\gamma : j'' \to j'''$ in $J_{i''}$ such that $X_{i''\gamma}$ coequalizes them. Then

$$(\beta, X_{i''\gamma} \cdot g) : (i', j') \to (i'', j''')$$

is a morphism of K coequalizing the given pair.

Now we have a functor $U : K \to \mathscr{E}$ which sends (i, j) to X_{ij} and (α, f) to f; we regard this as an object of Ind-$\mathscr{E}$. Since $U \cdot u_i = T(i)$ for each i, the functors u_i induce morphisms of ind-objects $\lambda_i : T(i) \to U$; we claim that these form a cone under the diagram T. For $(\lambda_i)_j$ is the equivalence class of the identity morphism $X_{ij} \to U(i, j)$, and clearly contains all those $f : X_{ij} \to U(i', j')$ for which $(\alpha, f) : (i, j) \to (i', j')$ is a morphism of K.

Finally, suppose we are given any cone $(\tau_i : T(i) \to W)_{i \in I}$ under T in Ind-$\mathscr{E}$. Each τ_i consists of a J_i-indexed family of equivalence clases of maps from X_{ij} into vertices of W. Putting these together, we obtain a K-indexed family of equivalence classes which is readily checked to be a morphism of ind-objects $\tau : U \to W$, and to be the unique factorization of $(\tau_i)_{i \in I}$ through $(\lambda_i)_{i \in I}$. So $(\lambda_i)_{i \in I}$ is a colimiting cone. □

It is clear that any functor $F : \mathscr{E} \to \mathscr{E}'$ between locally small categories can be extended to a functor Ind-F : Ind-$\mathscr{E} \to$ Ind-$\mathscr{E}'$, and that this extension is itself functorial in F. From the method of proof of Theorem 1.2, the following result is very nearly obvious.

Lemma 1.3. *For any F, the functor* Ind-F *preserves filtered colimits.*

Proof. The reason why this is not altogether obvious is that the definition of morphisms in the category K constructed in the proof of 1.2 involves the category $\mathscr{E}$, as well as the index categories I and J_i. If F is full and faithful, then the category K' constructed similarly but using $\mathscr{E}'$ instead of $\mathscr{E}$ is isomorphic to K; in general F induces an obvious functor $F_0 : K \to K'$, which is easily seen to be cofinal ([12], I 8.1.1) and to make the diagram

$$\begin{array}{ccc} K & \xrightarrow{U} & \mathscr{E} \\ \downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F} \\ K' & \xrightarrow{U'} & \mathscr{E}' \end{array}$$

commute. Hence Ind-$F(U)$ and U' are (canonically) isomorphic as objects of Ind-$\mathscr{E}'$. □

For a poset P, the embedding $\downarrow(-) : P \to \mathrm{Idl}(P)$ has a left adjoint iff P has directed joins (the adjoint necessarily sends an ideal to its join in P). A similar result holds for categories:

Lemma 1.4. *A locally small category $\mathscr{E}$ has (small) filtered colimits iff the embedding $y:\mathscr{E}\to\text{Ind-}\mathscr{E}$ has a left adjoint.*

Proof. If $\mathscr{E}$ has filtered colimits, then since

$$\begin{aligned}\text{Hom}_{\text{Ind-}\mathscr{E}}((X_i)_{i\in I}, y(Z)) &= \varprojlim_i \text{Hom}_{\mathscr{E}}(X_i, Z)\\ &\cong \text{Hom}_{\mathscr{E}}(\varinjlim_i X_i, Z)\end{aligned}$$

it is clear that y has a left adjoint which sends each ind-object to its colimit in $\mathscr{E}$. Conversely if y has a left adjoint L, then the same isomorphism shows that, for each ind-object $(X_i)_{i\in I}$, $L((X_i)_{i\in I})$ is a colimit for the X_i in $\mathscr{E}$. □

We shall denote the left adjoint of y, when it exists, by $\varinjlim$. Putting together the last two lemmas, we are now able to verify the third of the principles we invoked in defining the hom-sets of Ind-$\mathscr{E}$.

Proposition 1.5. (i) *For any object X of $\mathscr{E}$, the constant ind-object $y(X)$ is finitely-presentable in* Ind-$\mathscr{E}$.

(ii) *If idempotents split in $\mathscr{E}$, then every finitely-presentable object of* Ind-$\mathscr{E}$ *is isomorphic to a constant object.*

Proof. (i) By definition, we have

$$\text{Hom}_{\text{Ind-}\mathscr{E}}(y(X), (Y_j)_{j\in J}) = \varinjlim_j \text{Hom}_{\mathscr{E}}(X, Y_j);$$

so the functor $\text{Hom}_{\text{Ind-}\mathscr{E}}(y(X), -)$ may be factored as the composite

$$\text{Ind-}\mathscr{E} \xrightarrow{\text{Ind-}H} \text{Ind-}\mathscr{S} \xrightarrow{\varinjlim} \mathscr{S}$$

where H is the functor $\text{Hom}_{\mathscr{E}}(X, -)$. Now the first factor preserves filtered colimits by Lemma 1.3, and the second preserves all colimits since it is a left adjoint.

(ii) Conversely, suppose $(X_i)_{i\in I}$ is finitely-presentable in Ind-$\mathscr{E}$. Then since we have a filtered colimit

$$(X_i)_{i\in I} \cong \varinjlim_i y(X_i),$$

we can factor the identity map on $(X_i)_{i\in I}$ through one of the $y(X_i)$, and so express the former as a retract of the latter. Since y is full and faithful, the idempotent endomorphism of $y(X_i)$ corresponding to this retraction derives from an idempotent endomorphism of X_i in $\mathscr{E}$; on splitting this, we obtain an object of $\mathscr{E}$ whose image under y is isomorphic to $(X_i)_{i\in I}$. □

Thanks to Proposition 1.5, we can frequently recover a category $\mathscr{E}$ (up to equivalence) from Ind-$\mathscr{E}$ as its full subcategory of finitely-presentable objects. Once again, the analogue of this result for posets is well known; it is the fact that the principal ideals are exactly the 'compact' elements of Idl(P).

2. Continuous categories

We begin this section by recalling the concept which we wish to generalize. The notion of *continuous lattice* was introduced by Scott [33] and has been extensively studied [8]; more recently, attention has also been focused on *continuous posets* [28]. Both these concepts depend on the 'way-below' relation, which is definable in any poset with directed joins: we say a is *way below* b in such a poset P (and write $a \ll b$) if, whenever $S \subseteq P$ is directed and $\bigvee S \geq b$, there exists $s \in S$ with $s \geq a$. For any $a \in P$, we write $\downarrow(a)$ for the set $\{b \in P \mid b \ll a\}$; it is easy to verify that $\downarrow(a)$ is downwards closed, and closed under finite joins insofar as they exist in P.

We say a poset P is *continuous* if it has directed joins and, for every $a \in P$, the set $\downarrow(a)$ is directed and has join equal to a. (If P also has finite joins, and is thus a complete lattice, then the hypothesis "$\downarrow(a)$ is directed" is redundant.) For our purposes, a more useful characterization of continuous lattices is provided by:

Lemma 2.1. *Let P be a poset with directed joins. Then P is continuous iff the map $\bigvee : \mathrm{Idl}(P) \to P$ has a left adjoint.*

Proof. If the left adjoint exists, it must send $a \in P$ to the unique smallest ideal I with $\bigvee I \geq a$, i.e. to the intersection of all such ideals. But from the definition of $\ll$, it is clear that this intersection is precisely $\downarrow(a)$; so the existence of the left adjoint implies that $\downarrow(a)$ is an ideal (equivalently, directed). It is then clear that $a \leq \bigvee I$ implies $\downarrow(a) \subseteq I$; the reverse implication holds iff $a \leq \bigvee(\downarrow(a))$ – but since $b \ll a$ implies $b \leq a$, we always have $\bigvee(\downarrow(a)) \leq a$. So $\downarrow(-) : P \to \mathrm{Idl}(P)$ is left adjoint to $\bigvee$ iff P is continuous. □

We may now generalize the condition of Lemma 2.1 from posets to categories in an obvious way: we define a locally small category $\mathscr{C}$ to be *continuous* if it has (small) filtered colimits and the functor $\varinjlim : \text{Ind-}\mathscr{C} \to \mathscr{C}$ has a left adjoint. (In view of Lemma 1.1, we may thus interpret Lemma 2.1 as saying that a poset is a continuous category iff it is a continuous poset.)

Before investigating the consequences of this definition, we give a lemma which will be useful in many cases in verifying the existence of a left adjoint to $\varinjlim$.

Lemma 2.2. *Suppose that $\mathscr{C}$ has filtered colimits and pullbacks and that, for each $f : X \to Y$ in $\mathscr{C}$, the pullback functor $f^* : \mathscr{C}/Y \to \mathscr{C}/X$ preserves filtered colimits. Then the functor $\varinjlim : \text{Ind-}\mathscr{C} \to \mathscr{C}$ is a fibration (in the sense of* [11]).

Proof. Let $(Y_i)_{i \in I}$ be an ind-object with colimit Y, and $f : X \to Y$ a morphism of $\mathscr{C}$. Writing X_i for the pullback $X \times_Y Y_i$, we obtain an ind-object $(X_i)_{i \in I}$, which by hypothesis has colimit X; and the projections $X_i \to Y_i$ define a morphism of ind-objects $\bar{f} : (X_i)_{i \in I} \to (Y_i)_{i \in I}$ with $\varinjlim(\bar{f}) = f$. It is easy to verify that $\bar{f}$ is a cartesian morphism (with respect to the functor $\varinjlim$), and conversely that an arbitrary morphism of ind-objects is cartesian iff it factors as an isomorphism followed by a morphism of the form $\bar{f}$. Hence the cartesian morphisms of Ind-$\mathscr{E}$ are stable under composition, and so $\varinjlim$ is a fibration. □

Corollary 2.3. *Under the hypotheses of Lemma 2.2, constructing a left adjoint to* $\varinjlim : \text{Ind-}\mathscr{E} \to \mathscr{E}$ *is equivalent to constructing an initial object in each of its fibres.*

Proof. To construct the adjoint at a particular object X of $\mathscr{E}$, we have to find an initial object in the comma category $(X \downarrow \varinjlim)$. But the fact that $\varinjlim$ is a fibration easily implies that an initial object in the fibre over X, together with the identity map from X to its colimit, is initial in this comma category; the converse is obvious. □

We note that the hypotheses of 2.2 and 2.3 are satisfied either if $\mathscr{E}$ satisfies 'Axiom AB5' (finite limits commute with filtered colimits) or if $\mathscr{E}$ is a topos (in which case the functors f^* preserve all colimits).

The next result provides us with a plentiful supply of continuous categories.

Proposition 2.4. *For any locally small category* $\mathscr{E}$*, the category* Ind-$\mathscr{E}$ *is continuous.*

Proof. We already know Ind-$\mathscr{E}$ has filtered colimits (1.2). The embedding $y : \mathscr{E} \to \text{Ind-}\mathscr{E}$ induces a full embedding $\text{Ind-}y : \text{Ind-}\mathscr{E} \to \text{Ind-Ind-}\mathscr{E}$; we shall show that Ind-y is left adjoint to $\varinjlim : \text{Ind-Ind-}\mathscr{E} \to \text{Ind-}\mathscr{E}$. We have already observed that any ind-object $(X_i)_{i \in I}$ is the colimit in Ind-$\mathscr{E}$ of the objects $y(X_i)$, $i \in I$; i.e. the composite $\varinjlim \cdot \text{Ind-}y$ is (naturally) isomorphic to the identity. Let $(T_i)_{i \in I}$ be any filtered diagram in Ind-$\mathscr{E}$, and suppose $T = \varinjlim_i T_i$ is the ind-object $(X_j)_{j \in J}$. Then because each $y(X_j)$ is finitely-presentable in Ind-$\mathscr{E}$, the canonical maps $y(X_j) \to T$ in Ind-$\mathscr{E}$ each factor in an essentially unique way through some $T_i \to T$, and so we get a unique map in Ind-Ind-$\mathscr{E}$

$$(y(X_j))_{j \in J} \to (T_i)_{i \in I}$$

whose image under $\varinjlim$ is the identity map on T. It is straightforward to verify that this map is a component of a natural transformation from $\text{Ind-}y \cdot \varinjlim$ to the identity functor on Ind-Ind-$\mathscr{E}$, which satisfies the 'triangular identities' with the isomorphism $\mathrm{id}_{\text{Ind-}\mathscr{E}} \to \varinjlim \cdot \text{Ind-}y$. So we have an adjunction

$$\text{Ind-}y \dashv \varinjlim. \qquad \square$$

Corollary 2.5. *Any locally finitely presentable category (in the sense of Gabriel-Ulmer* [7]*) is continuous.*

Proof. It is well known that a locally finitely presentable category $\mathscr{E}$ is equivalent to the ind-completion of its full subcategory $\mathscr{E}_{\mathrm{fp}}$ of finitely-presentable objects. □

For our next result, we need an 'adjoint-lifting' lemma which seems not be widely known; so, although it was proved in [21], we repeat the statement of it here.

Lemma 2.6. *Suppose given a diagram of categories and functors*

$$\begin{array}{ccc} \mathscr{F}' & \xrightarrow{T'} & \mathscr{E}' \\ {\scriptstyle I'}\downarrow\uparrow{\scriptstyle R'} & & {\scriptstyle I}\downarrow\uparrow{\scriptstyle R} \\ \mathscr{F} & \xrightarrow{T} & \mathscr{E} \end{array}$$

in which $\mathscr{E}'$ is a (pseudo-)retract of $\mathscr{E}$ (i.e. $RI \cong \mathrm{id}_{\mathscr{E}'}$), $\mathscr{F}'$ is a retract of $\mathscr{F}$, and we have isomorphisms $TI' \cong IT'$, $RT \cong T'R'$ which are compatible with the retraction isomorphisms in the sense that

$$\begin{array}{ccc} RTI' & \xrightarrow{\sim} & T'R'I' \\ \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} \\ RIT' & \xrightarrow{\sim} & T' \end{array}$$

commutes. Suppose further that T has a left adjoint L, and that idempotents split in $\mathscr{F}'$. Then T' has a left adjoint.

Proof. See [21], Lemma 1.5. □

In general, the hypothesis that idempotents split in $\mathscr{F}'$ cannot be omitted; the 'naive' construction $L' = R'LI$ yields a functor which is not itself left adjoint to T', but which has an idempotent endomorphism whose image is the desired left adjoint. However, in the applications which concern us this restriction will not be irksome; for we shall be dealing with categories which possess filtered colimits, and the image of an idempotent may be computed as a colimit over the two-element monoid $\{1, e : e^2 = e\}$, which is a filtered category.

Proposition 2.7. *Let $\mathscr{E}$ be a continuous category, and let $\mathscr{E}'$ be a (pseduo-)retract of $\mathscr{E}$ (as in Lemma 2.6) by functors which preserve filtered colimits. Then $\mathscr{E}'$ is continuous.*

Proof. First, the hypotheses imply that $\mathscr{E}'$ has filtered colimits, since every filtered diagram in $\mathscr{E}'$ is in the essential image of the retraction $R : \mathscr{E} \to \mathscr{E}'$. Now we simply apply Lemma 2.6 to the diagram

$$\begin{array}{ccc} \text{Ind-}\mathscr{E}' & \xrightarrow{\varinjlim} & \mathscr{E}' \\ {\scriptstyle \text{Ind-}I}\downarrow\uparrow{\scriptstyle \text{Ind-}R} & & {\scriptstyle I}\downarrow\uparrow{\scriptstyle R} \\ \text{Ind-}\mathscr{E} & \xrightarrow{\varinjlim} & \mathscr{E} \end{array}$$

which commutes because I and R preserve filtered colimits. □

Theorem 2.8. *A locally small category $\mathscr{E}$ is continuous iff it is a retract of a category of the form* Ind-$\mathscr{F}$ *by functors preserving filtered colimits.*

Proof. If $\mathscr{E}$ is continuous, the functor $\varinjlim$ and its left adjoint L express it as a retract of Ind-$\mathscr{E}$ (note that the counit of the adjunction $(L \dashv \varinjlim)$ is necessarily an isomorphism, since the unit of $(\varinjlim \dashv y)$ is an isomorphism), and they both preserve colimits since they have right adjoints. The converse follows directly from Propositions 2.4 and 2.7. □

Of course, 2.8 generalizes the characterization of continuous posets as retracts of posets of the form Idl(P) ('algebraic posets') by maps preserving joins ('Scott-continuous maps'). (For continuous lattices, this result is already to be found in [33].) But it is worth noting that the proof of 2.8 actually tells us slightly more than is claimed in the statement; for it shows that an arbitrary continuous category can be embedded as a retract of one of the form Ind-$\mathscr{E}$ in such a way that the retraction is right adjoint to the inclusion. That is, if (ignoring problems of size) we write $\mathfrak{K}$ for the 2-category of categories of the form Ind-$\mathscr{E}$, functors preserving filtered colimits (which we might as well call 'Scott-continuous functors') and natural transformations, then the categories which we obtain by splitting (pseudo-)idempotents in $\mathfrak{K}$ (i.e. the continuous categories) may in fact all be obtained by splitting idempotent *comonads.* This observation will be of importance when we come to consider injective toposes in the next section.

It is natural to ask whether, in a continuous category $\mathscr{E}$, we have some analogue of the way-below relation in continuous posets. Indeed we do; but it turns out that we must regard it not as a 'relation' (i.e. a property of certain morphisms of $\mathscr{E}$), but as an additional structure which may be carried by such morphisms. We shall devote the rest of this section to developing it.

Let $\mathscr{E}$ be a continuous category, and write $L : \mathscr{E} \to \text{Ind-}\mathscr{E}$ for the left adjoint of $\varinjlim$. We define a *wavy arrow* from X to Y in $\mathscr{E}$ (denoted $X \rightsquigarrow Y$) to be a morphism $y(X) \to L(Y)$ in Ind-$\mathscr{E}$, i.e. an equivalence class of morphisms from X to vertices of the filtered diagram $L(Y)$. We write $\mathscr{H}om_{\mathscr{E}}(X, Y)$ (or simply $\mathscr{H}om(X, Y)$) for the set of all wavy arrows from X to Y.

Clearly, we have a canonical map

$$\varepsilon : \mathscr{H}om_{\mathscr{E}}(X, Y) \to \text{Hom}_{\mathscr{E}}(X, Y)$$

which sends the equivalence class of a morphism $X \to Y_i$ (Y_i some vertex of $L(Y)$) to the composite $X \to Y_i \to \varinjlim_i Y_i \cong Y$. (Thus ε is just the functor $\varinjlim$ applied to morphisms $y(X) \to L(Y)$ in Ind-$\mathscr{E}$.) We shall call $\varepsilon(f)$ the *underlying straight arrow* of the wavy arrow f. However, ε is not in general a monomorphism; two different wavy arrows may have the same underlying straight arrow, which is why we must regard 'waviness' as a structure rather than a property.

Example 2.9. Let $\mathscr{E}$ be the category of (left) G-sets, where G is a group. Then $\mathscr{E}$ is locally finitely presentable and so continuous by Corollary 2.5; moreover, from the proof of Proposition 2.4 we see that $L(X)$, for a G-set X, is the filtered diagram whose vertices correspond to all morphisms from (a representative set of) finitely-presentable G-sets to X. Now it is easy to see that a G-set X is finitely-presentable iff (a) X has finitely many G-orbits and (b) for each $x \in X$, the stabilizer subgroup $G_x = \{g \in G \mid gx = x\}$ is finitely-generated. (We are indebted to R. Börger for this observation.) Suppose G itself is not finitely-generated. Since G is finitely-presentable as a G-set, the two projections $G \times G \to G$ both define wavy arrows $G \times G \rightsquigarrow 1$ in $\mathscr{E}$; and these wavy arrows are distinct since the two projections cannot be coequalized by any map from G to a finitely-presentable G-set. But they have the same underlying straight arrow, since there is only one map $G \times G \to 1$.

Since y and L are functors, it is clear that we can compose a wavy arrow $f: X \rightsquigarrow Y$ in $\mathscr{E}$ with either a straight arrow $g: T \to X$ or a straight arrow $h: Y \to Z$ (the results being wavy arrows $T \rightsquigarrow Y$ and $X \rightsquigarrow Z$ respectively), by forming the composites $f \cdot y(g)$ and $L(h) \cdot f$ in Ind-$\mathscr{E}$. Moreover, since Ind-$\mathscr{E}$ is a category it is clear that these two types of composition are associative and commute with each other; and since $\varinjlim$ is a functor the map ε converts both types of composition into the ordinary composition of straight arrows. Thus we have proved:

Lemma 2.10. *The assignment* $(X, Y) \mapsto \mathscr{W}\!om(X, Y)$ *is a profunctor* (=*distributeur*) $\mathscr{E} \dashrightarrow \mathscr{E}$, *i.e. a functor* $\mathscr{E}^{\mathrm{op}} \times \mathscr{E} \to \mathscr{S}$; *and* ε *is a morphism of profunctors from* $\mathscr{W}\!om$ *to the unit* (*Yoneda*) *profunctor* Hom : $\mathscr{E} \dashrightarrow \mathscr{E}$. □

A slightly less trivial, but very useful, result is:

Lemma 2.11. *Let* $f: X \rightsquigarrow Y$ *and* $g: Y \rightsquigarrow Z$ *be two wavy arrows in* $\mathscr{E}$. *Then the composites* $g \cdot \varepsilon(f)$ *and* $\varepsilon(g) \cdot f$ *are equal as wavy arrows* $X \rightsquigarrow Z$.

Proof. Regarding the composites as morphisms $y(X) \to L(Z)$ in Ind-$\mathscr{E}$, it is easy to see that each is the composite

$$y(X) \xrightarrow{\ f\ } L(Y) \xrightarrow{\ i\ } y(Y) \xrightarrow{\ g\ } L(Z),$$

where $i: L(Y) \to y(Y)$ is the unique morphism of ind-objects lying over the identity map on Y. □

Lemma 2.11 tells us that we have a well-defined composition for wavy arrows; and it follows easily from the definitions that the eight possible associative laws

$$(f\cdot g)\cdot h = f\cdot (g\cdot h),$$

where each of the arrows f, g, h may be either wavy or straight, are all satisfied. In particular, taking g to be straight and the other two to be wavy, we would be entitled to regard composition of wavy arrows as defining a morphism of profunctors

$$\mu : \mathscr{H}om \otimes_{\mathscr{E}} \mathscr{H}om \to \mathscr{H}om,$$

if only the domain of this morphism were legitimately definable. The trouble is that, since $\mathscr{E}$ is in general a large category, $\mathscr{H}om \otimes_{\mathscr{E}} \mathscr{H}om(X, Y)$ is defined as a quotient of the proper class

$$\coprod_{Z\in \mathrm{ob}\,\mathscr{E}} \mathscr{H}om(Z, Y)\times \mathscr{H}om(X, Z),$$

and so we have no right to regard it as a set. However, we observe that every wavy arrow $[f] : Z \rightsquigarrow Y$ can be factored as

$$Z \xrightarrow{\ f\ } Y_i \overset{[\mathrm{id}]}{\rightsquigarrow} Y$$

for some vertex Y_i of $L(Y)$, and so any composable pair $X \rightsquigarrow Z \rightsquigarrow Y$ is equal (as a member of $\mathscr{H}om \otimes_{\mathscr{E}} \mathscr{H}om(X, Y)$) to one of the form $X \rightsquigarrow Y_i \rightsquigarrow Y$; thus in the definition of $\mathscr{H}om \otimes_{\mathscr{E}} \mathscr{H}om(X, Y)$ we may restrict the variable Z to run over the *set* of objects Y_i which occur as vertices of the diagram $L(Y)$. (More explicitly, we may define $\mathscr{H}om \otimes_{\mathscr{E}} \mathscr{H}om(X, Y)$ to be $\varinjlim_i \mathscr{H}om(X, Y_i)$.)

In this way $\mathscr{H}om \otimes_{\mathscr{E}} \mathscr{H}om$ becomes a legitimate profunctor, and we may regard composition of wavy arrows as a morphism of profunctors μ as above. Moreover, one of the eight associative laws tells us that μ is itself associative in an obvious sense.

Proposition 2.12. *The morphism of profunctors $\mu : \mathscr{H}om \otimes_{\mathscr{E}} \mathscr{H}om \to \mathscr{H}om$ defined above is an isomorphism.*

Proof. First we show that μ is surjective, i.e. that any wavy arrow can be factored as a composite of two wavy arrows. (The argument here generalizes the proof of the well-known 'interpolation property' of the way-below relation in a continuous poset – cf. [8], Theorem I 1.18.) Let X be an object of $\mathscr{E}$; write $(X_i)_{i\in I}$ for the ind-object $L(X)$ and $(X_{ij})_{j\in J_i}$ for each $L(X_i)$. Since L is a functor, the $L(X_i)$ form a filtered diagram in Ind-$\mathscr{E}$, and by the proof of Theorem 1.2 the colimit of this diagram is a filtered diagram in $\mathscr{E}$ whose vertices are all the X_{ij}, $j\in \coprod_{i\in I} J_i$. Now the functor $\varinjlim : \text{Ind-}\mathscr{E} \to \mathscr{E}$ preserves colimits, and so

$$\varinjlim(\varinjlim_i L(X_i)) \cong \varinjlim_i(\varinjlim L(X_i)) \cong \varinjlim_i X_i \cong X.$$

Transposing this isomorphism, we get a morphism of ind-objects

$$L(X) \to \varinjlim_i L(X_i)$$

lying over the identity map on X; in particular for each $i \in I$ the canonical map $\lambda_i : X_i \to X$ can be factored as

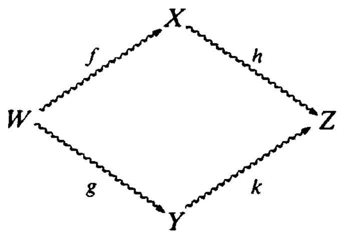
$$X_i \xrightarrow{\ h\ } X_{i'j} \longrightarrow X_{i'} \xrightarrow{\ \lambda_{i'}\ } X$$

for some i' and some j. Furthermore, the composite $X_i \to X_{i'j} \to X_{i'}$ represents the ith component of a map of ind-objects $L(X) \to L(X)$ over X, which must be the identity; so this composite represents the same wavy arrow $X_i \rightsquigarrow X$ as id_{X_i}. Thus, given any wavy arrow $Y \rightsquigarrow X$ represented by $f : Y \to X_i$, say), we may factor it as

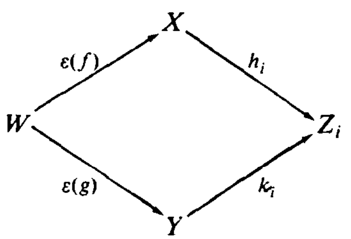
$$Y \overset{[hf]}{\rightsquigarrow} X_{i'} \overset{[\mathrm{id}]}{\rightsquigarrow} X,$$

as required.

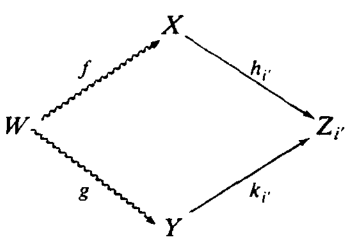
Next we must show that μ is injective, i.e. that if



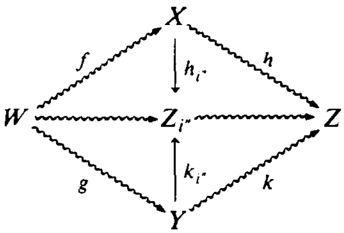
is a commutative square of wavy arrows, then the pairs (h, f) and (k, g) are already equal in $\mathcal{H}om \otimes_{\mathscr{E}} \mathcal{H}om(W, Z)$. First, since $L(Z)$ is a filtered diagram, we may represent h and k by morphisms $h_i : X \to Z_i$ and $k_i : Y \to Z_i$ for the same index i. Now the square



need not commute, but, since both ways round represent the same wavy arrow $W \rightsquigarrow Z$, we can find $i \to i'$ in the index category for $L(Z)$ such that $Z_i \to Z_{i'}$ coequalizes them. Even so, the square



need not commute at the wavy level; but since the two wavy arrows $W \rightsquigarrow Z_{i'}$ have the same underlying straight arrow, it follows from Lemma 2.11 that they have equal composites with (the underlying straight arrow of) any wavy arrow with domain $Z_{i'}$. Accordingly, we now use the first part of the proof to factor $[\mathrm{id}_{Z_{i'}}] : Z_{i'} \rightsquigarrow Z$ as a composite $Z_{i'} \rightsquigarrow Z_{i''} \rightsquigarrow Z$, and replace $h_{i'}$ and $k_{i'}$ by their composites with the underlying straight arrow of $Z_{i'} \rightsquigarrow Z_{i''}$. We then have a diagram



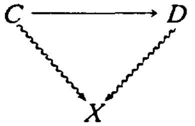
in which all cells commute at the wavy level, from which we deduce that (h, f) and (k, g) are equal in $\mathcal{H}om \otimes_{\mathcal{E}} \mathcal{H}om(W, Z)$. $\square$

In view of Proposition 2.12, we may consider the inverse of μ as a morphism of profunctors $\mathcal{H}om \to \mathcal{H}om \otimes_{\mathcal{E}} \mathcal{H}om$. Since μ is associative, μ^{-1} is coassociative; and from the way in which μ was defined it is easy to see that $\varepsilon : \mathcal{H}om \to \mathrm{Hom}$ is a counit for μ^{-1}. We thus have:

Theorem 2.13. *For any continuous category $\mathcal{E}$, the structure $(\mathcal{H}om, \mu^{-1}, \varepsilon)$ is an idempotent profunctor comonad.* $\square$

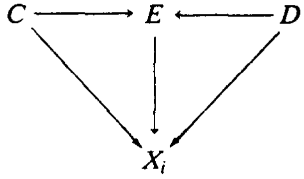
So far, we have not imposed any 'size restrictions' on $\mathcal{E}$ beyond that of local smallness. However, it frequently happens in practice that a continuous category $\mathcal{E}$, though not itself small, has a small generating subcategory of a particularly nice kind. We next investigate this possibility.

Let $\mathcal{C}$ be a small full subcategory of $\mathcal{E}$. We shall say that $\mathcal{C}$ is *$\mathcal{E}$-filtered* if, for every object X of $\mathcal{E}$, the comma category $\mathcal{C}/X$ (whose objects are $\mathcal{E}$-morphisms with domain in $\mathcal{C}$ and codomain X) is filtered. Note that this condition holds if $\mathcal{C}$ has finite colimits which are preserved by the inclusion $\mathcal{C} \to \mathcal{E}$; but it is not necessary to assume that $\mathcal{C}$ has finite colimits. We shall write $\mathcal{C}\wr X$ for the category whose objects are all wavy arrows from objects of $\mathcal{C}$ to X, and whose morphisms are commutative triangles of the form

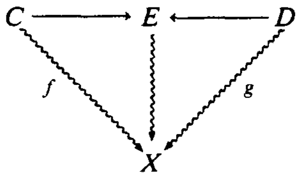


Lemma 2.14. *If $\mathcal{C}$ is $\mathcal{E}$-filtered, then $\mathcal{C}\wr X$ is filtered for every X.*

Proof. Let $L(X)=(X_i)_{i\in I}$. First, $\mathscr{C}\int X$ is nonempty since the filtered category I is nonempty and $\mathscr{C}/X_i$ is nonempty for any $i\in I$. Next, suppose we have two wavy arrows $f:C\rightsquigarrow X$, $g:D\rightsquigarrow X$. Since I is filtered, we can represent f and g by straight arrows into the same X_i, and then use filteredness of $\mathscr{C}/X_i$ to construct as diagram



with E in $\mathscr{C}$, which we can interpret as a diagram



The verification of the third condition for filteredness is similar. $\square$

We recall that a full subcategory $\mathscr{C}$ of $\mathscr{E}$ is said to be *dense* ([25], p. 241) if every object of $\mathscr{E}$ can be expressed as a colimit of objects of $\mathscr{C}$. Of course, if such an expression exists, there is a canonical one: we can express X as the colimit of the forgetful functor $U_X:\mathscr{C}/X\to\mathscr{E}$ which sends $(f:C\to X)$ to C.

Lemma 2.15. *If $\mathscr{C}$ is dense in $\mathscr{E}$, then any object X of $\mathscr{C}$ is expressible as the colimit of the forgetful functor $\mathscr{U}_X:\mathscr{C}\int X\to\mathscr{E}$.*

Proof. Consider a cone λ under the diagram $\mathscr{U}_X$ (with vertex Y, say). For each vertex X_i of the diagram $L(X)$, we may construct a cone λ_i under U_{X_i}; specifically, if $f:C\to X_i$, we define $(\lambda_i)_f$ to be $\lambda_{[f]}$, where $[f]$ is the wavy arrow $C\rightsquigarrow X$ represented by f. So by density of $\mathscr{C}$ we obtain unique factorizations $v_i:X_i\to Y$ of each of these cones through the colimiting ones. From the uniqueness, it is clear that the v_i themselves form a cone under $L(X)$, and so define a unique map $v:\varinjlim L(X)\cong X\to Y$. So the canonical cone under $\mathscr{U}_X$ with vertex X is a colimiting cone. $\square$

Suppose now that $\mathscr{C}$ is both dense and $\mathscr{E}$-filtered. Then for any object X of $\mathscr{E}$, we can regard $\mathscr{U}_X$ as an object of Ind-$\mathscr{E}$ with colimit X. Moreover, from the definition of wavy arrows it is clear that there is a morphism of ind-objects from $\mathscr{U}_X$ to $L(X)$, whose fth component (for $f:C\rightsquigarrow X$ an object of $\mathscr{C}\int X$) is f itself, and which lies over the identity morphism on X. But by the universal property of $L(X)$, we must also have a unique morphism of ind-objects $L(X) \to \mathcal{U}_X$ over the identity on X, and the composite $L(X) \to \mathcal{U}_X \to L(X)$ must be the identity. Consider the composite $\mathcal{U}_X \to L(X) \to \mathcal{U}_X$. If $f : C \rightsquigarrow X$ is any object of $\mathscr{C} \int X$, then by the first half of the proof of Proposition 2.12 we may factor f as $C \overset{h}{\rightsquigarrow} D \overset{g}{\rightsquigarrow} X$ (where there is clearly no loss of generality in supposing that D is an object of $\mathscr{C}$); and then $\varepsilon(h) : f \to g$ is a morphism of $\mathscr{C} \int X$, so that if $k : D \to \mathcal{U}_X (l : E \rightsquigarrow X)$ represents the gth component of the above composite, then $k \cdot \varepsilon(h)$ represents its fth component. But the diagram

$$\begin{array}{ccc} D & \xrightarrow{\quad k \quad} & E \\ & {\scriptstyle \varepsilon(g)} \searrow \qquad \swarrow {\scriptstyle \varepsilon(l)} & \\ & X & \end{array}$$

must commute since this morphism lies over the identity on X; hence

$$\begin{array}{ccc} C & \xrightarrow{\quad k \cdot \varepsilon(h) \quad} & E \\ & {\scriptstyle f} \searrow \qquad \swarrow {\scriptstyle l} & \\ & X & \end{array}$$

commutes at the wavy level, i.e. $k \cdot \varepsilon(h)$ is a morphism of $\mathscr{C} \int X$. But this means that the given endomorphism of $\mathcal{U}_X$ is the identity, and so $\mathcal{U}_X$ is isomorphic to $L(X)$ in Ind-$\mathscr{E}$. Furthermore, it is not hard to see that this isomorphism is natural in X, if we make $X \mapsto \mathcal{U}_X$ into a functor $\mathscr{E} \to$ Ind-$\mathscr{E}$ in the obvious way; and so we have proved:

Proposition 2.16. *Let $\mathscr{E}$ be a continuous category. Then the left adjoint $L : \mathscr{E} \to$ Ind-$\mathscr{E}$ of $\varinjlim$ may be taken to factor through* Ind-$\mathscr{C} \hookrightarrow$ Ind-$\mathscr{E}$, *where $\mathscr{C}$ is any small, full, dense, $\mathscr{E}$-filtered subcategory of $\mathscr{E}$.* ☐

In view of 2.16, we obtain a refinement of Theorem 2.8:

Corollary 2.17. *The following conditions on a category $\mathscr{E}$ are equivalent:*

(i) *$\mathscr{E}$ is a retract, by filtered-colimit-preserving functors, of a category of the form* Ind-$\mathscr{C}$ *where $\mathscr{C}$ is small.*

(ii) *$\mathscr{E}$ is continuous and has a small, full, dense, $\mathscr{E}$-filtered subcategory.*

Proof. (ii)$\Rightarrow$(i): If $\mathscr{C}$ is such a subcategory, then the functors

$$\mathscr{E} \xrightarrow{\quad L \quad} \text{Ind-}\mathscr{C} \quad \text{and} \quad \text{Ind-}\mathscr{C} \longrightarrow \text{Ind-}\mathscr{E} \xrightarrow{\quad \varinjlim \quad} \mathscr{E}$$

express $\mathscr{E}$ as a retract of Ind-$\mathscr{C}$; and they preserve filtered colimits since the inclusion Ind-$\mathscr{C} \to$ Ind-$\mathscr{E}$ is the ind-extension (in the sense of Lemma 1.3) of $\mathscr{C} \to \mathscr{E}$.

(i)$\Rightarrow$(ii): It is easy to see that $\mathscr{C}$ is dense and Ind-$\mathscr{C}$-filtered in Ind-$\mathscr{C}$; and its image under a retraction Ind-$\mathscr{C} \to \mathscr{E}$ has the same properties relative to $\mathscr{E}$. ☐

As we noted in the case of Theorem 2.8, the first half of the proof above actually tells us rather more than is claimed in the statement: namely that any $\mathscr{E}$ satisfying (ii) is embeddable as a coreflective subcategory of some Ind-$\mathscr{C}$. But for a small category $\mathscr{C}$, there is no harm in identifying Ind-$\mathscr{C}$ with its image in the functor category $[\mathscr{C}^{op}, \mathscr{S}]$ (since every object of this category has a *canonical* representation as a small colimit of representables); as usual, we shall call a functor $\mathscr{C}^{op} \to \mathscr{S}$ *flat* if it is (isomorphic to) a filtered colimit of representables, and write Flat$(\mathscr{C}^{op}, \mathscr{S})$ for the full subcategory of flat functors. (It is well known [4] that if $\mathscr{C}$ has finite colimits, then the flat functors $\mathscr{C}^{op} \to \mathscr{S}$ are just the finite-limit-preserving ones.)

If $\mathscr{C}$ is a subcategory of a continuous category $\mathscr{E}$ as in 2.17(ii), it is naturally of interest to have a characterization of the coreflective subcategory of Flat$(\mathscr{C}^{op}, \mathscr{S})$ which is the image of $\mathscr{E}$ under this identification. Of course, the flat functor $\mathscr{C}^{op} \to \mathscr{S}$ which corresponds to the ind-object $\mathscr{W}_X$ is just (the restriction to $\mathscr{C}$ of) the functor $\mathscr{H}om(-, X)$.

Proposition 2.18. *With $\mathscr{C}$ and $\mathscr{E}$ as in Corollary 2.17, a flat functor $F : \mathscr{C}^{op} \to \mathscr{S}$ is isomorphic to one of the form $\mathscr{H}om(-, X)$ (X an object of $\mathscr{E}$) iff it satisfies the following condition:*

(*) *For every object C of $\mathscr{C}$ and every $x \in F(C)$, there exists a wavy arrow $f : C \rightsquigarrow D$ (with D an object of $\mathscr{C}$) and $y \in F(D)$ such that $x = F(\varepsilon f)(y)$.*

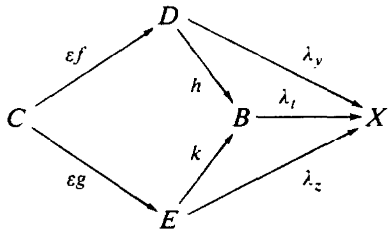
Proof. If F is the functor $\mathscr{H}om(-, X)$, then condition (*) follows from the 'subdivisibility' of wavy arrows, i.e. the first half of the proof of Proposition 2.12. (As we have already remarked, there is no loss of generality in requiring the object in the middle of the factorization to lie in the subcategory $\mathscr{C}$.) Conversely, suppose (*) is satisfied; then (identifying F with an ind-object in $\mathscr{C}$) we wish to show that the counit map $\beta : L(\varinjlim F) \to F$ is an isomorphism.

Let $(C_i)_{i \in I}$ be the ind-object corresponding to F (note that the indices i are just the elements of $\coprod_{C \in ob\,\mathscr{C}} F(C)$), and let X be its colimit in $\mathscr{E}$. The map β is defined as follows: given a wavy arrow $f : C \rightsquigarrow X$, choose a representative $h : C \to C_i$ for the fth component of the unique morphism of ind-objects $\mathscr{W}_X \to (C_i)_{i \in I}$ over X, and then define $\beta_C(f) = F(h)(i)$ (where we regard the index i as an element of $F(C_i)$). It is straightforward to verify that this is well defined, and a natural transformation of functors.

It is easy to see that β is surjective; for if $x \in F(C)$, then by (*) we can find $f : C \rightsquigarrow D$ and $y \in F(D)$ mapping onto x, and then the composite

$$C \overset{f}{\rightsquigarrow} D \xrightarrow{\lambda_y} X$$

is an element of $\mathscr{H}om(C, X)$ which is mapped by β to x. (Here λ_y denotes the yth component of the colimiting cone.) But in fact the above construction yields a well-defined natural transformation $\alpha : F \to \mathscr{H}om(-, X)$, which is a one-sided inverse for β; to see this, it is sufficient to prove that it is well defined, since naturality is then obvious. Suppose we have $f: C \rightsquigarrow D$, $g: C \rightsquigarrow E$, $y \in F(D)$ and $z \in F(E)$ such that $F(\varepsilon f)(y) = x = F(\varepsilon g)(z)$; then by flatness of F we can find morphisms $h: D \to B$, $k: E \to B$ in $\mathscr{C}$ and $t \in F(B)$ such that



commutes. In particular, the composites $h \cdot f$ and $k \cdot g: C \rightsquigarrow B$ have the same underlying straight arrow; but by the argument already given to prove surjectivity of β, λ_t underlies some wavy arrow $B \rightsquigarrow X$, and so by Lemma 2.11 the composites $\lambda_t \cdot h \cdot f = \lambda_y \cdot f$ and $\lambda_t \cdot k \cdot g = \lambda_z \cdot g$ are equal as wavy arrows. Thus $\alpha(x)$ is a well-defined wavy arrow $C \rightsquigarrow X$.

So we have expressed the functor F as a retract of $\mathscr{H}om(-, X)$ in the category of flat functors $\mathscr{C}^{\mathrm{op}} \to \mathscr{S}$ whose corresponding ind-object has colimit X in $\mathscr{E}$; but since $\mathscr{H}om(-, X)$ is initial in this category, it has no proper retracts, and so is isomorphic to F. □

Proposition 2.18 tells us that a continuous category $\mathscr{E}$ can be reconstructed (up to equivalence) from the pair $(\mathscr{C}, T)$, where $\mathscr{C}$ is a generating subcategory of $\mathscr{E}$ as in 2.17, and $T: \mathscr{C} \dashrightarrow \mathscr{C}$ is the profunctor obtained by restricting $\mathscr{H}om: \mathscr{E} \dashrightarrow \mathscr{E}$. As we have already remarked, the proof of idempotency of $\mathscr{H}om$ which we gave in Proposition 2.12 remains valid if we restrict the objects involved to lie in $\mathscr{C}$; so T is still an idempotent profunctor comonad. Moreover, T is *left flat* in the terminology of [24], i.e. the functors $T(-, C): \mathscr{C}^{\mathrm{op}} \to \mathscr{S}$ are all flat. (Equivalently, the functor $(-) \otimes_{\mathscr{C}} T: [\mathscr{C}, \mathscr{S}] \to [\mathscr{C}, \mathscr{S}]$ preserves finite limits.)

In the converse direction, note that a left flat profunctor $T: \mathscr{C} \dashrightarrow \mathscr{D}$ between small categories is essentially the same thing as a functor $\mathscr{D} \to \mathrm{Flat}(\mathscr{C}^{\mathrm{op}}, \mathscr{S}) \simeq \text{Ind-}\mathscr{C}$, and hence essentially the same as a filtered-colimit-preserving functor $\text{Ind-}\mathscr{D} \to \text{Ind-}\mathscr{C}$. So the bicategory $\mathfrak{K}_0$ of small categories, left flat profunctors and morphisms of profunctors is equivalent (contravariantly at the level of 1-arrows) to a full subcategory of the 2-category $\mathfrak{K}$ considered after Theorem 2.8, namely that whose objects have the form Ind-$\mathscr{C}$ where $\mathscr{C}$ is small. Hence if we split the idempotents (or more particularly, the idempotent comonads) in $\mathfrak{K}_0$, we obtain a full subcategory of the idempotent-completion of $\mathfrak{K}$, namely the 2-category of continuous categories satisfying the size restriction of 2.17. The passage from $(\mathscr{C}, T)$ to the subcategory of flat functors satisfying (∗) is clearly functorial, and extends the passage from $\mathscr{C}$ to Ind-$\mathscr{C}$, so it is the required embedding of the idempotent-completion of $\mathfrak{K}_0$ in that of $\mathfrak{K}$.

A curious side-effect of Proposition 2.18 is to tell us that a left flat, idempotent profunctor comonad T on a small category $\mathscr{C}$ is determined up to isomorphism by its image under the counit map $\varepsilon: T \to \mathrm{Hom}_{\mathscr{C}}$; for in order to state the condition (*) we need only know which straight arrows of $\mathscr{C}$ underlie wavy arrows, and not what the wavy arrows themselves are. It is not at all clear *ab initio* why this should be so.

3. Injective toposes revisited

In an earlier paper [21], the first author investigated a notion of injectivity for Grothendieck toposes (more generally, for bounded $\mathscr{S}$-toposes, where $\mathscr{S}$ is an arbitrary base topos). In that study, the structure of idempotent profunctor comonad played an important rôle, for reasons which were not entirely clear. The appearance of the same structure in our investigation of continuous categories is the key which enables us to open up the link between the two concepts, generalizing the link which Scott [33] discovered between injective spaces and continuous lattices, and incidentally clarifying the status of the two conditions which appeared in [21] as unwarranted assumptions.

We shall continue to assume for notational purposes that our base category $\mathscr{S}$ is 'the' topos of constant sets, but in practice it could easily be generalized to any topos with a natural number object, by rewriting our arguments (which are all constructive) in the language of categories indexed over $\mathscr{S}$ [31].

First we recall one of the main results of [21]:

Proposition 3.1. *A bounded $\mathscr{S}$-topos is injective (with respect to sheaf subtopos inclusions) iff it is a retract in $\mathfrak{BTop}/\mathscr{S}$ of a functor category $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ where $\mathscr{C}$ has finite limits.* □

It will be convenient for the time being to broaden our considerations to include all retracts in $\mathfrak{BTop}/\mathscr{S}$ of presheaf toposes; we shall call them *quasi-injective*. (It is not clear whether there is in fact any injectivity condition which characterizes these toposes; it is interesting to note that Hoffmann [14] has characterized the corresponding class of spaces by a projectivity condition.)

Proposition 3.2. *Let $\mathscr{F}$ be a quasi-injective topos. Then $\mathscr{F}$ has enough points, and its category of points is continuous and satisfies the size restriction of Corollary 2.17.*

Proof. Since $\mathscr{F}$ is a retract of some $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$, it is in particular a surjective image of $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$; but presheaf toposes always have enough points. Now the category of points of a Grothendieck topos has filtered ($\mathscr{S}$-indexed) colimits by [20], Corollary 7.14; and from the proof of that fact, it is easily deduced that these colimits are preserved by the functors $\mathfrak{BTop}/\mathscr{S}(\mathscr{S}, \mathscr{E}) \to \mathfrak{BTop}/\mathscr{S}(\mathscr{S}, \mathscr{F})$ induced by geometric morphisms $\mathscr{E} \to \mathscr{F}$. So the category of points of $\mathscr{F}$ is a retract, by filtered-colimit-preserving functors, of $\mathfrak{BTop}/\mathcal{S}(\mathcal{S},[\mathcal{C}^{op},\mathcal{S}])$; but the latter is equivalent by Diaconescu's theorem [4] to $\mathrm{Flat}(\mathcal{C},\mathcal{S})$ – i.e. to Ind-$\mathcal{C}^{op}$. □

We note in passing that the Löwenheim–Skolem theorem for points of Grothendieck toposes ([20], Theorem 7.16) ensures that if $\mathcal{E}$ is the category of points of an arbitrary Grothendieck topos, then it has a small, full, dense, $\mathcal{E}$-filtered subcategory.

In the converse direction, let $\mathcal{E}$ be a continuous category satisfying the hypotheses of 2.17. We wish to construct a quasi-injective topos $\mathcal{F}$ whose category of points is equivalent to $\mathcal{E}$. Although we shall see eventually that $\mathcal{F}$ may be constructed directly from $\mathcal{E}$, in order to establish its basic properties we shall need to work in terms of a particular generating subcategory $\mathcal{C}$ of $\mathcal{E}$ as in 2.17. Let T, as before, denote the restriction to $\mathcal{C}$ of the profunctor $\mathcal{H}om$ on $\mathcal{E}$. Since T is left flat, we can regard $(-)\otimes_{\mathcal{C}}T$ as the inverse image of a geometric morphism $t:[\mathcal{C},\mathcal{S}]\to[\mathcal{C},\mathcal{S}]$ over $\mathcal{S}$ (which is of course idempotent, since T is).

We shall also wish to refer to a particular Grothendieck topology J_T on $\mathcal{C}^{op}$ determined by T, as follows: a cosieve on an object C of $\mathcal{C}$ is J_T-covering iff it contains the underlying straight arrows of all wavy arrows with domain C. The fact that J_T is a Grothendieck topology follows easily from the known properties of T; in particular, the 'local character' axiom ($T2$) of [12], II 1.1 is implied by the idempotency of T.

Proposition 3.3. *Let $\mathcal{C}$ be a small category, and T a left flat, idempotent profunctor comonad on $\mathcal{C}$. Then the image (in the topos-theoretic sense) of the geometric morphism $t:[\mathcal{C},\mathcal{S}]\to[\mathcal{C},\mathcal{S}]$ induced by T is* $\mathrm{Shv}(\mathcal{C}^{op},J_T)$, *where J_T is the Grothendieck topology defined above. Moreover,* $\mathrm{Shv}(\mathcal{C}^{op},J_T)$ *is also the image of t in the idempotent-splitting sense; in particular, it is a quasi-injective topos.*

Proof. To identify the image of t, we have to determine which subobjects $R\rightarrowtail \mathrm{Hom}_{\mathcal{C}}(C,-)$ (i.e. which cosieves on C) are mapped to isomorphisms by the functor $t^*=(-)\otimes_{\mathcal{C}}T$. But

$$R\otimes_{\mathcal{C}}T\rightarrowtail \mathrm{Hom}_{\mathcal{C}}(C,-)\otimes_{\mathcal{C}}T\cong T(C,-)$$

is an isomorphism iff, for each wavy arrow $f:C\rightsquigarrow D$, there exists $g:C\to E$ in R and $h:E\rightsquigarrow D$ such that $h\cdot g=f$. Clearly this condition implies that $\varepsilon(f)$ is in R for every such f; but conversely if R contains all the $\varepsilon(f)$, then we may factor f as a composite $C\overset{g}{\rightsquigarrow}E\overset{h}{\rightsquigarrow}D$ and then $\varepsilon(g)\in R$. So the covering sieves on C are precisely the J_T-covering ones.

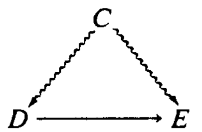
Now let

$$[\mathcal{C},\mathcal{S}]\xrightarrow{\;r\;}\mathrm{Shv}(\mathcal{C}^{op},J_T)\xrightarrow{\;i\;}[\mathcal{C},\mathcal{S}]$$

denote the (topos-theoretic) image factorization of t; to show that it is also a splitting of the idempotent t, we must show that ri is isomorphic to the identity map on $\mathrm{Shv}(\mathscr{C}^{\mathrm{op}}, J_T)$, or equivalently that if F is a J_T-sheaf then $F \cong t_*(F)$. But $t_*(F) = T \pitchfork_{\mathscr{C}} F$, where $T \pitchfork_{\mathscr{C}} (-)$ denotes the right adjoint of $(-) \otimes_{\mathscr{C}} T$, from which we readily deduce the formula

$$t_*(F)(C) = \varprojlim_{f : C \rightsquigarrow D} F(D),$$

the inverse limit being taken over the category whose objects are all wavy arrows with domain C, and whose morphisms are commutative triangles of the form



Now the assertion that F is a J_T-sheaf tells us that the canonical map

$$F(C) \to \varprojlim_{(f : C \to D) \in R} F(D)$$

is an isomorphism, where R is the minimal J_T-covering cosieve on C, i.e. the category whose objects are all underlying straight arrows of wavy arrows with domain C. It is easy to see that this inverse limit maps monomorphically into the one above, i.e. that the canonical natural transformation $F \to t_*(F)$ is mono. To show it is an isomorphism, we need to show that if $x = (x_f)_{f : C \rightsquigarrow D}$ is any element of $\varprojlim_{f : C \rightsquigarrow D} F(D)$, and f, g are two wavy arrows with the same underlying straight arrow, then we must have $x_f = x_g$. But f and g are coequalized by any wavy arrow with domain D, and so x_f and x_g must have the same image in $t_*(F)(D) = \varprojlim_{D \rightsquigarrow E} F(E)$. Hence by what we have already proved, $x_f = x_g$; i.e. x is in the image of the canonical map $F(C) \to t_*(F)(C)$. $\square$

Remark 3.4. In the case when $\mathscr{C}$ has finite colimits, Proposition 3.3 was proved in [21] under the additional hypothesis that the 'underlying straight arrow' map ε was a monomorphism. In view of Example 2.9 and the proof above, it now appears that this additional assumption was unjustified; however, without it we cannot characterize the topologies J_T on $\mathscr{C}^{\mathrm{op}}$ which arise from profunctors T as in 3.3, as simply as we did in [21], Lemma 2.3. (Conditions (i) and (ii) of the characterization given there remain valid, but (iii) holds only for products and not for arbitrary pullbacks, and there does not seem to be any simple way of reconstructing T from J_T.)

Proposition 3.5. *Let $\mathscr{E}$ be a continuous category satisfying the size restriction of Corollary* 2.17. *Then there exists a quasi-injective topos $\mathscr{F}$ whose category of points is equivalent to $\mathscr{E}$. If in addition $\mathscr{E}$ has finite colimits* (*and is thus cocomplete*), *then $\mathscr{F}$ may be taken to be injective.*

Proof. Let $\mathscr{C}$ be a small, full, dense, $\mathscr{E}$-filtered subcategory of $\mathscr{E}$, and let T denote the restriction to $\mathscr{C}$ of the profunctor $\mathscr{H}\!om$ on $\mathscr{E}$. Define $\mathscr{F}$ to be $\mathrm{Shv}(\mathscr{C}^{\mathrm{op}}, J_T)$, where J_T is constructed from T as in 3.3. Then $\mathscr{F}$ is quasi-injective by 3.3, and by [20], Proposition 7.13, its points correspond to flat functors $\mathscr{C}^{\mathrm{op}} \to \mathscr{S}$ which are 'continuous' for the topology J_T, i.e. send J_T-covering sieves to epimorphic families. But since every object of $\mathscr{C}$ has a smallest J_T-covering cosieve (viz. the set of all underlying straight arrows of wavy arrows with the given object as domain), it is sufficient to check the continuity condition for these minimal sieves – and for them, it is precisely the condition (∗) of Proposition 2.18. So the equivalence $\mathscr{E} \simeq \mathfrak{BTop}/\mathscr{S}(\mathscr{S}, \mathscr{F})$ follows directly from 2.18.

In the special case when $\mathscr{E}$ has finite colimits, we may choose our generating subcategory $\mathscr{C}$ to be closed under finite colimits in $\mathscr{E}$ (in which case it is certainly $\mathscr{E}$-filtered, as we remarked earlier); then $[\mathscr{C}, \mathscr{S}]$ is injective by [21], Proposition 1.2, and hence so is its retract $\mathscr{F}$. □

To complete the circle, it remains to show that a quasi-injective topos is determined up to equivalence by its category of points. But we already know this fact for presheaf toposes; for we have $\mathfrak{BTop}/\mathscr{S}(\mathscr{S}, [\mathscr{C}, \mathscr{S}]) \simeq \text{Ind-}\mathscr{C}$, and by 1.5 we can recover $\mathscr{C}$ (or at least its idempotent-completion, which is sufficient to determine the functor category $[\mathscr{C}, \mathscr{S}]$) from Ind-$\mathscr{C}$ as the full subcategory of finitely-presentable objects. And this result extends easily to retracts:

Theorem 3.6. *The functor $\mathscr{F} \mapsto \mathfrak{BTop}/\mathscr{S}(\mathscr{S}, \mathscr{F})$ is an equivalence of 2-categories between the full subcategory of $\mathfrak{BTop}/\mathscr{S}$ consisting of quasi-injective toposes, and the 2-category $\mathfrak{Cont}$ of continuous categories satisfying the hypotheses of 2.17 and Scott-continuous (i.e. filtered-colimit-preserving) functors between them.*

Proof. If we restrict to presheaf toposes and to continuous categories of the form Ind-$\mathscr{C}$, then we have an equivalence (the inverse functor being described above). And it is straightforward to verify that any equivalence between 2-categories extends (essentially uniquely) to an equivalence between their idempotent-completions. □

Corollary 3.7. *Any quasi-injective topos is expressible as the image of an idempotent comonad on a presheaf topos. In particular, any injective topos is expressible as the image of an idempotent comonad on a presheaf topos $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ where $\mathscr{C}$ has finite limits.*

Proof. We know that the corresponding assertion holds in $\mathfrak{Cont}$, by the remarks after Theorem 2.8 and Corollary 2.17; so we may transfer it across the equivalence of Theorem 3.6. The particular case follows from the general one as in the proof of 3.5, since the category of points of an injective topos is cocomplete ([21], Corollary 1.7). □

Corollary 3.7 tells us that the first of the two unsupported assumptions which were made in Section 2 of [21], that we could restrict our atttention to idempotent profunctor comonads, was in fact justified, although (as we have seen) the second was not.

There remains one question of interest: we have seen that a continuous category $\mathscr{E}$ (satisfying the conditions of 2.17) determines a quasi-injective topos $\mathscr{F}$ up to equivalence, but the only method we know for constructing $\mathscr{F}$ involves making an arbitrary choice of a generating subcategory of $\mathscr{E}$. Can we construct $\mathscr{F}$ directly from $\mathscr{E}$, without making any such choices? The answer is yes; but is not immediately apparent from the nature of the construction that it always yields a topos, which is why we chose to give a more roundabout, but more explicit, construction first.

Proposition 3.8. *Let $\mathscr{E}$ be a continuous category satisfying the hypotheses of* 2.17. *Then the full subcategory $\mathfrak{Cont}(\mathscr{E}, \mathscr{S})$ of the functor category $[\mathscr{E}, \mathscr{S}]$ whose objects are Scott-continuous functors is a quasi-injective topos, and its category of points is equivalent to $\mathscr{E}$.*

Proof. First we note that $\mathfrak{Cont}(\mathscr{E}, \mathscr{S})$ is closed in $[\mathscr{E}, \mathscr{S}]$ under finite limits and arbitrary (small) colimits; so any Scott-continuous functor $f: \mathscr{E} \to \mathscr{E}'$ induces a functor $f^*: \mathfrak{Cont}(\mathscr{E}', \mathscr{S}) \to \mathfrak{Cont}(\mathscr{E}, \mathscr{S})$ which preserves finite limits and all colimits. In particular, if the domain and codomain of f^* are Grothendieck toposes, then it is the inverse image of a geometric morphism. But in the case when $\mathscr{E} = \text{Ind-}\mathscr{C}$, it is clear that a functor defined on $\mathscr{E}$ is Scott-continuous iff it is isomorphic to the ind-extension of its restriction to $\mathscr{C}$, and so $\mathfrak{Cont}(\mathscr{E}, \mathscr{S}) \simeq [\mathscr{C}, \mathscr{S}]$ is a (quasi-injective) Grothendieck topos whose category of points is equivalent to $\mathscr{E}$. The result for a general $\mathscr{E}$ now follows from the fact that any functor preserves images of idempotents. □

4. Exponentiable toposes

As indicated in the Introduction, our main objective in this paper is to characterize the exponentiable objects in the 2-category $\mathfrak{BTop}/\mathscr{S}$ of bounded $\mathscr{S}$-toposes (where we shall continue to assume for notational purposes that $\mathscr{S}$ is the topos of constant sets). Given toposes $\mathscr{E}$ and $\mathscr{F}$ (bounded over $\mathscr{S}$), we shall say that the exponential $\mathscr{F}^{\mathscr{E}}$ exists if the category-valued functor

$$\mathfrak{BTop}/\mathscr{S}((-) \times_{\mathscr{S}} \mathscr{E}, \mathscr{F})$$

is representable (in the up-to-equivalence sense), the representing object being denoted $\mathscr{F}^{\mathscr{E}}$. We say $\mathscr{E}$ is *exponentiable* if $\mathscr{F}^{\mathscr{E}}$ exists for all $\mathscr{F}$; the operation $(-)^{\mathscr{E}}$ is then a (pseudo-)functor $\mathfrak{BTop}/\mathscr{S} \to \mathfrak{BTop}/\mathscr{S}$, right pseudo-adjoint to $(-) \times_{\mathscr{S}} \mathscr{E}$. Similarly if we keep $\mathscr{F}$ fixed and allow $\mathscr{E}$ to vary, we obtain a contravariant functor $\mathscr{F}^{(-)}$ defined on the full subcategory of exponentiable toposes in $\mathfrak{BTop}/\mathscr{S}$.

We shall make frequent use of the well-known equivalence between bounded $\mathscr{S}$-toposes and first-order geometric theories in the laguange of $\mathscr{S}$ (see [27]). As an example, we begin with a simple but useful lemma:

Lemma 4.1. *For any small category $\mathscr{C}$, the functor category $[\mathscr{C}, \mathscr{S}]$ is exponentiable in $\mathfrak{BTop}/\mathscr{S}$.*

Proof. For any $\mathscr{S}$-topos $\mathscr{E}$, the pullback $\mathscr{E} \times_{\mathscr{S}} [\mathscr{C}, \mathscr{S}]$ is equivalent to $[\mathscr{C}, \mathscr{E}]$ by Diaconescu's theorem [4]. But by Wraith's characterization of lax colimits in $\mathfrak{Top}$ [36], $[\mathscr{C}, \mathscr{E}]$ is the tensor of $\mathscr{E}$ with $\mathscr{C}$ in $\mathfrak{Top}$; that is, we have

$$\mathfrak{BTop}/\mathscr{S}([\mathscr{C}, \mathscr{E}], \mathscr{F}) = [\mathscr{C}, \mathfrak{BTop}/\mathscr{S}(\mathscr{E}, \mathscr{F})]$$

for any $\mathscr{F}$. Accordingly, if $\mathscr{F}$ is the classifying topos for a geometric theory $\mathbb{T}$, we may define $\mathscr{F}^{[\mathscr{C}, \mathscr{S}]}$ to be the classifying topos for the theory $[\mathscr{C}, \mathbb{T}]$ whose models are diagrams of type $\mathscr{C}$ in the category of $\mathbb{T}$-models. (It is straightforward to construct a presentation for this theory from one for $\mathbb{T}$; see [20], Example 6.55(v).) □

Lemma 4.2. *Exponentiability is a local property; that is,*

(i) *if $\mathscr{E}$ is exponentiable then so is $\mathscr{E}/X$ for any X, and*

(ii) *if $\mathscr{E}/X$ is exponentiable and X has global support in $\mathscr{E}$, then $\mathscr{E}$ is exponentiable.*

Proof. (i) By a special case of Lemma 4.1, we know that $\mathscr{E}/X$ is exponentiable in $\mathfrak{BTop}/\mathscr{E}$. But a standard argument on exponentials (cf. [20], Exercise 1.8) shows that $(f : \mathscr{F} \to \mathscr{E})$ is exponentiable in $\mathfrak{BTop}/\mathscr{E}$ iff the pullback functor along f has a right adjoint

$$\Pi_f : \mathfrak{BTop}/\mathscr{F} \to \mathfrak{BTop}/\mathscr{E}.$$

Since the latter condition is clearly stable under composition of bounded geometric morphisms, it follows at once that if $\mathscr{E}$ is exponentiable in $\mathfrak{BTop}/\mathscr{S}$ then so is $\mathscr{E}/X$.

(ii) If X has global support in $\mathscr{E}$, then the diagram

$$\mathscr{E}/X \times X \rightrightarrows \mathscr{E}/X \to \mathscr{E}$$

is a universal coequalizer diagram in $\mathfrak{BTop}/\mathscr{S}$ (this follows from the descent theorem for open surjections [25], but can in fact be proved much more simply). So the exponential $\mathscr{F}^{\mathscr{E}}$, if it exists, should be the equalizer of the diagram

$$\mathscr{F}^{\mathscr{E}/X} \rightrightarrows \mathscr{F}^{\mathscr{E}/X \times X}.$$

But exponentiability of $\mathscr{E}/X$ implies exponentiability of $\mathscr{E}/X \times X$, by the first part; hence we can simply define $\mathscr{F}^{\mathscr{E}}$ to be this equalizer. □

Lemma 4.3. *A retract (in $\mathfrak{BTop}/\mathscr{S}$) of exponentiable topos is exponentiable.*

Proof. Suppose $\mathscr{E}$ is a retract of an exponentiable topos $\mathscr{E}'$. The idempotent endomorphism of $\mathscr{E}'$ whose image is $\mathscr{E}$ induces, for any $\mathscr{F}$, an idempotent endomorphism of $\mathscr{F}^{\mathscr{E}'}$. If we split this idempotent (which we may do since $\mathfrak{BTop}/\mathscr{S}$ has finite limits), we obtain the exponential $\mathscr{F}^{\mathscr{E}}$. □

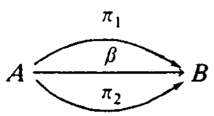
In [17], M. Hyland showed that a locale X is exponentiable (in the category of locales) iff the exponential S^X exists, where S is the Sierpiński locale. As we indicated in the Introduction, the underlying reason for this is that S is the free object on one generator in the opposite of the category of locales [18]; in $\mathfrak{BTop}/\mathscr{S}$, the corresponding role is played by the object classifier $\mathscr{S}[X]$ [24]. We now embark on the proof that existence of $\mathscr{S}[X]^{\mathscr{E}}$ implies exponentiability of $\mathscr{E}$; for technical reasons it will be convenient to divide it into two stages.

Lemma 4.4. *Let $\mathscr{E}$ be a topos for which the exponential $\mathscr{S}[X]^{\mathscr{E}}$ exists. Then for any small category $\mathscr{C}$, the exponential $\mathscr{S}[\mathbb{C}]^{\mathscr{E}}$ exists, where $\mathscr{S}[\mathbb{C}]$ is the classifying topos for the theory of diagrams of type $\mathscr{C}$.*

Proof. By Lemma 4.1, we can regard $\mathscr{S}[\mathbb{C}]$ as the exponential $\mathscr{S}[X]^{[\mathscr{C},\mathscr{S}]}$; and general exponential nonsense shows that $(\mathscr{S}[X]^{[\mathscr{C},\mathscr{S}]})^{\mathscr{E}}$, if it exists, should be equivalent to $(\mathscr{S}[X]^{\mathscr{E}})^{[\mathscr{C},\mathscr{S}]}$. But the latter topos exists by another application of 4.1. □

Theorem 4.5. *A topos $\mathscr{E}$ is exponentiable in $\mathfrak{BTop}/\mathscr{S}$ iff the exponential $\mathscr{S}[X]^{\mathscr{E}}$ exists.*

Proof. let $\mathscr{F}$ be an arbitrary bounded $\mathscr{S}$-topos, and suppose it classifies a geometric theory $\mathbb{T}$. Using the techniques of [35], we may present $\mathbb{T}$ in such a way that its models appear as diagrams of a certain type $\mathscr{C}$ satisfying certain axioms, each of which says that a particular morphism constructed 'geometrically' from the diagram is an isomorphism. (For example if $\mathbb{T}$ is the theory of objects with a single binary operation β, we may present it as the theory of diagrams of type



subject to the axiom that $(\pi_1, \pi_2): A \to B \times B$ is an isomorphism.) Now each such geometric construction, when applied to the generic diagram of type $\mathscr{C}$, gives rise to a geometric morphism

$$\mathscr{S}[\mathbb{C}] \to \mathscr{S}[2]$$

where $\mathscr{S}[2]$ is the morphism classifier over $\mathscr{S}$; and the statement that the construction applied to $\mathbb{T}$-models yields an isomorphism means that the composite

$$\mathscr{F} \to \mathscr{S}[\mathbb{C}] \to \mathscr{S}[2]$$

factors (up to isomorphism) through the 'diagonal' inclusion $\mathscr{S}[X] \to \mathscr{S}[2]$ which classifies the identity map on the generic object. More particularly, we have a pullback diagram in $\mathfrak{BTop}/\mathscr{S}$ of the form

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \Pi_A \mathcal{S}[X] \\ \downarrow & & \downarrow \\ \mathcal{S}[\mathbb{C}] & \longrightarrow & \Pi_A \mathcal{S}[2] \end{array}$$

where A is an index set for the axioms in the given presentation of $\mathbb{T}$. But the toposes $\mathcal{S}[\mathbb{C}]$, $\Pi_A \mathcal{S}[X]$ and $\Pi_A \mathcal{S}[2]$ all classify theories of diagrams of a certain type (with no additional axioms); so by Lemma 4.4 we can exponentiate each of them to the power $\mathcal{E}$ provided $\mathcal{S}[X]^{\mathcal{E}}$ exists. Also, the functor $(-)^{\mathcal{E}}$, being a right adjoint, should preserve pullbacks; so we may now define $\mathcal{F}^{\mathcal{E}}$ to be the pullback of

$$\begin{array}{ccc} & & (\Pi_A \mathcal{S}[X])^{\mathcal{E}} \\ & & \downarrow \\ \mathcal{S}[\mathbb{C}]^{\mathcal{E}} & \longrightarrow & (\Pi_A \mathcal{S}[2])^{\mathcal{E}} \end{array}$$

and verify that it has the required universal property. □

So far in this section we have not invoked the theory of continuous categories or of injective toposes. But, as we indicated in the Introduction, there is a very simple link between these topics and exponentiability:

Lemma 4.6. *If $\mathcal{E}$ is exponentiable in $\mathfrak{BTop}/\mathcal{S}$, then it is a continuous category.*

Proof. From the definition of the theory it classifies, it is clear that the object classifier $\mathcal{S}[X]$ is injective in the sense considered in Section 3. Also, the functor $(-)\times_{\mathcal{S}}\mathcal{E}$ preserves inclusions (cf. [20], Proposition 4.47), so its right adjoint $(-)^{\mathcal{E}}$ preserves injectives; thus $\mathcal{S}[X]^{\mathcal{E}}$, if it exists, is an injective topos. But we have

$$\mathcal{E} \cong \mathfrak{BTop}/\mathcal{S}(\mathcal{E}, \mathcal{S}[X]) \simeq \mathfrak{BTop}/\mathcal{S}(\mathcal{S}, \mathcal{S}[X]^{\mathcal{E}});$$

thus $\mathcal{E}$ is equivalent to the category of points of an injective topos. The result as stated follows from Proposition 3.2. □

In the converse direction, suppose $\mathcal{E}$ is a Grothendieck topos which is a continuous category. Then $\mathcal{E}$ has a small dense subcategory, which we may take to be closed under finite colimits and therefore $\mathcal{E}$-filtered; so it satisfies the size restriction of Corollary 2.17, and by Proposition 3.5 it is thus equivalent to the category of points of a (uniquely determined) injective topos $\mathcal{F}$. Clearly, $\mathcal{F}$ is the only possible candidate for the exponential $\mathcal{S}[X]^{\mathcal{E}}$; to prove that it is the exponential, we have to extend the known equivalence $\mathfrak{BTop}/\mathcal{S}(\mathcal{S}, \mathcal{F}) \simeq \mathcal{E}$ into a natural equivalence

$$\mathfrak{BTop}/\mathcal{S}(\mathcal{S}', \mathcal{F}) \simeq \mathcal{S}' \times_{\mathcal{S}} \mathcal{E}$$

for all bounded $\mathcal{S}$-toposes $(\gamma : \mathcal{S}' \to \mathcal{S})$.

Before embarking on this, let us fix some notation. $\mathscr{C}$ will denote a small (full) generating subcategory of $\mathscr{E}$, which we shall assume closed under finite colimits *and* limits in $\mathscr{E}$ (so that in particular $\mathscr{C}$ is a pretopos ([20], Definition 7.38) with arbitrary coequalizers). On $\mathscr{C}$ we have the Grothendieck topology K induced by the inclusion $\mathscr{C} \to \mathscr{E}$ (i.e. the collection of all sieves in $\mathscr{C}$ which are universally effective-epimorphic in $\mathscr{E}$), and also the left flat profunctor T obtained by restricting $\mathscr{H}om$ on $\mathscr{E}$ (from which we obtain a topology J_T on $\mathscr{C}^{op}$ as in 3.3). Now we have two different ways of embedding $\mathscr{E}$ into Ind-$\mathscr{C} \simeq \mathrm{Flat}(\mathscr{C}^{op}, \mathscr{S})$; the first sends an object X to the K-sheaf $\mathrm{Hom}(-,X)$, and the second sends X to the J_T-continuous functor $\mathscr{H}om(-,X)$. The two embeddings are respectively right and left adjoint to the functor $\varinjlim : \text{Ind-}\mathscr{C} \to \mathscr{E}$. (There is an interesting parallel between the existence of these two alternative representations of $\mathscr{E}$ and the confusion in the early days of sheaf theory ([9], pp. 4–5) about whether sheaves should be defined in terms of sections over open sets or closed sets.)

To establish the desired natural equivalence, we have to show that the equivalence between K-sheaves and J_T-continuous flat functors remains valid when we replace the category $\mathscr{C}$ and the topologies K and J_T by their 'pullbacks' along a (bounded) geometric morphism $\gamma : \mathscr{S}' \to \mathscr{S}$. Now the pullback $(\gamma^*\mathscr{C}, \tilde{\gamma}K)$ of a site $(\mathscr{C}, K)$ along a geometric morphism γ was described in detail in [22], in the particular case when $\mathscr{C}$ is a semilattice; the general case does not involve any additional complications other than notational ones. In particular, although it is in general not possible to describe $\tilde{\gamma}K$ explicitly in terms of K, we note that if K is the topology generated by a given (pullback-stable) family of sieves on $\mathscr{C}$, then $\tilde{\gamma}K$ is generated by the image of this family under γ^*. Thus it follows at once from the definition of J_T that we have $\tilde{\gamma}(J_T) = J_{(\gamma^*T)}$.

Next, we need to describe how the topology K may be generated from the profunctor T. First, since $\mathscr{C}$ is closed under finite colimits in $\mathscr{E}$, we know that K contains the precanonical (finite-cover) topology P on $\mathscr{C}$. Given this information, it now suffices to say which *filtered* sieves on objects C of $\mathscr{C}$ (i.e. filtered full subcategories of $\mathscr{C}/C$) are K-covering, since an arbitrary sieve R on C is covering iff the filtered sieve

$$\{f : D \to C \mid \exists \text{ a finite cover } \{g_i\}_i \text{ of } D \text{ s.t. each } f \cdot g_i \in R\}$$

is covering. But from the definition of wavy arrows, every filtered K-covering sieve on C contains the sieve

$$M_C = \{f : D \to C \mid \exists g : D \rightsquigarrow C \text{ with } \varepsilon g = f\}$$

which is itself K-covering since $\mathscr{E}$ is a continuous category. (Note that M_C itself is not necessarily filtered, if ε fails to be a monomorphism, but this does not matter for our purposes.) We thus conclude:

Lemma 4.7. *The topology K on $\mathscr{C}$ is generated by the precanonical topology P and the sieves M_C, $C \in \mathrm{ob}\,\mathscr{C}$; in particular a presheaf on $\mathscr{C}$ is a K-sheaf iff it is a P-sheaf and satisfies the sheaf axiom for the sieves M_C.*

Proof. The first statement follows immediately from the remarks above. For the second, we have to show in addition that a presheaf which satisfies the sheaf axiom for the M_C also satisfies the axiom for their pullbacks along morphisms of $\mathscr{C}$. But this is obvious, since if $f: D \to C$ is such a morphism then we have $M_D \subseteq f^*M_C$. □

Now the finite cover topology on a pretopos is preserved by inverse image functors (cf. [22], Lemma 4.2); so it follows from Lemma 4.7 that $\tilde{\gamma}K$ may be generated from γ^*T in the same way that K is generated from T. And the assertions "$\mathscr{C}$ is a pretopos with arbitrary coequalizers" and "T is a left flat, idempotent profunctor comonad on $\mathscr{C}$" are both expressible in geometric language, and so preserved by inverse image functors. So we are reduced to proving the following statement:

> "Given a pretopos $\mathscr{C}$ with arbitrary coequalizers and a left flat, idempotent profunctor comonad T on $\mathscr{C}$, let K and J_T be the topologies on $\mathscr{C}$ and $\mathscr{C}^{op}$ constructed from T as in 4.7 and 3.3 respectively. Then the category of K-sheaves on $\mathscr{C}$ is equivalent to the category of flat, J_T-continuous functors $\mathscr{C}^{op} \to \mathscr{S}$."

Unfortunately, this statement does not appear to be true in general, because the hypotheses are not sufficient to ensure that K-sheaves are flat functors on $\mathscr{C}^{op}$. The embedding of $\mathscr{C}$ into $\mathrm{Shv}(\mathscr{C}, P)$ (and hence the canonical functor $\mathscr{C} \to \mathrm{Shv}(\mathscr{C}, K)$) preserves coequalizers of equivalence relations – this follows from the fact that each regular epimorphism in $\mathscr{C}$ generates a P-covering sieve – but in order to ensure that arbitrary coequalizers are preserved, we need the additional information that each equivalence relation in $\mathscr{C}$ is covered by the finite powers of any (reflexive, symmetric) relation which generates it. Since such a cover is necessarily filtered, giving this information about our topology K is equivalent to giving the following information about T:

(★) Suppose S is a reflexive, symmetric relation on an object C of $\mathscr{C}$, and let $R = \bigcup_{n \geq 1} S^n$ be its equivalence closure. Then every wavy arrow $D \rightsquigarrow R$ factors through some finite power S^n of S.

Thus the pair $(\mathscr{C}, T)$ satisfies the condition (★) iff the canonical functor $\mathscr{C} \to \mathrm{Shv}(\mathscr{C}, K)$ preserves all coequalizers, iff $\mathrm{Shv}(\mathscr{C}, K)$ is contained in (and therefore a reflective subcategory of) $\mathrm{Flat}(\mathscr{C}^{op}, \mathscr{S})$. In particular if $(\mathscr{C}, T)$ is derived from a continuous topos $\mathscr{E}$ as originally envisaged, then condition (★) is satisfied.

Lemma 4.8. *Let $\mathscr{C}$ be an internal pretopos with arbitrary coequalizers in a topos $\mathscr{S}$, and T a profunctor on $\mathscr{C}$ satisfying (★). Then for any geometric morphism $\gamma: \mathscr{S}' \to \mathscr{S}$, the pair $(\gamma^*\mathscr{C}, \gamma^*T)$ satisfies (★).*

Proof. The statement of (★) is expressible (internally) in geometric language, except for the reference to equivalence closures. But since inverse image functors preserve natural number objects, they also preserve equivalence closures whenever these exist (as they do in a pretopos with coequalizers). □

We are now ready for the key step in the proof of our main theorem.

Lemma 4.9. *Let $\mathscr{C}$ be a pretopos with arbitrary coequalizers, and let T be a left flat, idempotent profunctor comonad on $\mathscr{C}$ satisfying (★). Let K and J_T be the topologies on $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ constructed from T as in* 4.7 *and* 3.3 *respectively. Then* $\mathrm{Shv}(\mathscr{C},K)$ *is equivalent to the category of points of* $\mathrm{Shv}(\mathscr{C}^{\mathrm{op}},J_T)$.

Proof. As indicated above, the assumption (★) ensures that $\mathrm{Shv}(\mathscr{C},K)$ is a reflective subcategory of $\mathrm{Flat}(\mathscr{C}^{\mathrm{op}},\mathscr{S})$. The category of points of $\mathrm{Shv}(\mathscr{C}^{\mathrm{op}},J_T)$ may also be regarded as a full subcategory of $\mathrm{Flat}(\mathscr{C}^{\mathrm{op}},\mathscr{S})$, namely the category of J_T-continuous functors. Since T is left flat, the functor $T\otimes_{\mathscr{C}}(-):[\mathscr{C}^{\mathrm{op}},\mathscr{S}]\to[\mathscr{C}^{\mathrm{op}},\mathscr{S}]$ maps $\mathrm{Flat}(\mathscr{C}^{\mathrm{op}},\mathscr{S})$ into itself; and it inherits an idempotent comonad structure from T. By Proposition 3.3, we know that the image of T on $\mathrm{Flat}(\mathscr{C}^{\mathrm{op}},\mathscr{S})$ is exactly the subcategory of J_T-continuous functors, since it corresponds to the image of the geometric endomorphism of $[\mathscr{C},\mathscr{S}]$ induced by $(-)\otimes_{\mathscr{C}} T$.

As a functor on $[\mathscr{C}^{\mathrm{op}},\mathscr{S}]$, $T\otimes_{\mathscr{C}}(-)$ has a right adjoint $T\pitchfork_{\mathscr{C}}(-)$, which may be defined by

$$T\pitchfork_{\mathscr{C}} F(C)=\varprojlim_{(f:D\rightsquigarrow C)} F(D).$$

Clearly, $T\pitchfork_{\mathscr{C}}(-)$ has an idempotent monad structure, and its image consists precisely of those presheaves on $\mathscr{C}$ which satisfy the sheaf axiom for the sieves M_C defined before Lemma 4.7. But it follows from left flatness of T that $T\pitchfork_{\mathscr{C}}(-)$ preserves sheaves for the precanonical topology; so its image on Flat $(\mathscr{C}^{\mathrm{op}},\mathscr{S})$ is just the category of K-sheaves. Now it is easy to see that the adjunction between $T\otimes_{\mathscr{C}}(-)$ and $T\pitchfork_{\mathscr{C}}(-)$ (as functors from $\mathrm{Flat}(\mathscr{C}^{\mathrm{op}},\mathscr{S})$ to itself) is itself idempotent, and hence that it restricts to an equivalence between $\mathrm{Shv}(\mathscr{C},K)$ and the category of J_T-continuous functors. □

At last we are ready to put together all the ingredients.

Theorem 4.10. *A bounded $\mathscr{S}$-topos $\mathscr{E}$ is exponentiable in $\mathfrak{BTop}/\mathscr{S}$ iff it is a continuous category.*

Proof. One direction is Lemma 4.6. Conversely, if $\mathscr{E}$ is a continuous category, it suffices by Theorem 4.5 to construct the exponential $\mathscr{S}[X]^{\mathscr{E}}$. We define this exponential to be $\mathrm{Shv}(\mathscr{C}^{\mathrm{op}},J_T)$, where $\mathscr{C}$ is a generating subcategory of $\mathscr{E}$, closed under finite limits and colimits, and T is the restriction to $\mathscr{C}$ of the profunctor $\mathscr{H}om$ on $\mathscr{E}$. Now let $(\gamma:\mathscr{S}'\to\mathscr{S})$ be an arbitrary bounded $\mathscr{S}$-topos. Working in the context of $\mathscr{S}'$-indexed categories, we have equivalences

$$\mathscr{S}' \times_{\mathscr{S}} \mathscr{E} \simeq \mathrm{Shv}(\gamma^*\mathscr{C}, \tilde{\gamma}K)$$

$$\simeq \mathfrak{BTop}/\mathscr{S}'(\mathscr{S}', \mathrm{Shv}(\gamma^*\mathscr{C}^{\mathrm{op}}, \tilde{\gamma}J_T)) \quad \text{by Lemma 4.9}$$

$$\simeq \mathfrak{BTop}/\mathscr{S}'(\mathscr{S}', \mathscr{S}' \times_{\mathscr{S}} \mathrm{Shv}(\mathscr{C}^{\mathrm{op}}, J_T))$$

from which we deduce

$$\mathfrak{BTop}/\mathscr{S}(\mathscr{S}' \times_{\mathscr{S}} \mathscr{E}, \mathscr{S}[X]) \simeq \mathfrak{BTop}/\mathscr{S}(\mathscr{S}', \mathrm{Shv}(\mathscr{C}^{\mathrm{op}}, J_T)).$$

So $\mathrm{Shv}(\mathscr{C}^{\mathrm{op}}, J_T)$ has the required universal property. □

5. Local compactness and exponentiability

In view of the main theorem of the last section, it is clearly of interest to find conditions on a site of definition for a topos $\mathscr{E}$ which are equivalent to $\mathscr{E}$ being a continuous category. In this section we tackle the problem in the particular case when $\mathscr{E}$ is localic (i.e. generated by subobjects of 1); it seems likely that some of our methods will extend to more general sites (using the techniques of [5]), but we shall not pursue the matter here.

Somewhat surprisingly, in view of the close formal similarity between our main theorem and Hyland's characterization of exponentiable locales, it turns out that not every exponentiable (=locally compact) locale generates an exponentiable topos of sheaves. However, there is a clear implication in the other direction:

Lemma 5.1. *Let $\mathscr{E}$ be an exponentiable topos. Then for any object X of $\mathscr{E}$, the lattice of subobjects of X is continuous.*

Proof. If $\mathscr{E}$ is exponentiable, then so is $\mathscr{E}/X$ by Lemma 4.2; in particular the exponential $[2,\mathscr{S}]^{\mathscr{E}/X}$ exists, where $[2,\mathscr{S}]$ is the Sierpiński topos over $\mathscr{S}$ [20, 4.37(iii)]. But $[2,\mathscr{S}]$ is a classifying topos for subobjects of 1 in $\mathscr{S}$-toposes; so it is easily seen to be injective, and hence $[2,\mathscr{S}]^{\mathscr{E}/X}$ in injective by the argument used in proving Lemma 4.6. But the category of points of this topos is equivalent to the poset of subobjects of X in $\mathscr{E}$; so the latter is a continuous poset, and hence (since it is a lattice in any case) a continuous lattice. □

Specializing to the case when $\mathscr{E}$ is the topos of sheaves on a sober space X, and taking the object X in the lemma to be the terminal object of $\mathscr{E}$, we deduce that if $\mathscr{E}$ is exponentiable then the open-set lattice of X must be continuous – equivalently [1, 15], X must be locally compact.

However, exponentiability of $\mathrm{Shv}(X)$ implies more than local compactness of X. To explain why, we need to introduce a strengthening of the way-below relation: we shall write $U \lll V$ (for U, V open subsets of X) if, for every filtered diagram $(F_i)_{i \in I}$ of sheaves on X with $\varinjlim_i F_i \cong V$, there exists $i \in I$ such that F_i has a section over U. If $\mathrm{Shv}(X)$ is a continuous category, this is equivalent to saying that there is a wavy arrow $U \rightsquigarrow V$ in Shv(X), since $L(V)$ (if it exists) is initial in the category of ind-objects with colimit V.

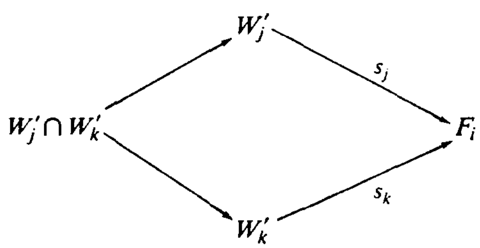
The definition of $\lll$ just given is unsatisfactory in that it refers to arbitrary (filtered diagrams of) sheaves on X, and not just to the open-set lattice of X. Later on, we shall give a characterization of $\lll$ entirely in terms of open sets; for the present, we note merely that it implies the relation $\ll$ (take the F_i to be a directed family of subobjects of 1), and that in an important special case this implication can be reversed. We recall that a locale is said to be *stably locally compact* [23] if it is a continuous lattice and, in addition, the way-below relation is stable under binary meets – i.e. $U_1 \ll V_1$ and $U_2 \ll V_2$ together imply $U_1 \cap U_2 \ll V_1 \cap V_2$. Examples of stably locally compact locales include all coherent locales and their retracts, and all locally compact regular locales.

Propostion 5.2. *In a stably locally compact locale, $U \ll V$ implies $U \lll V$.*

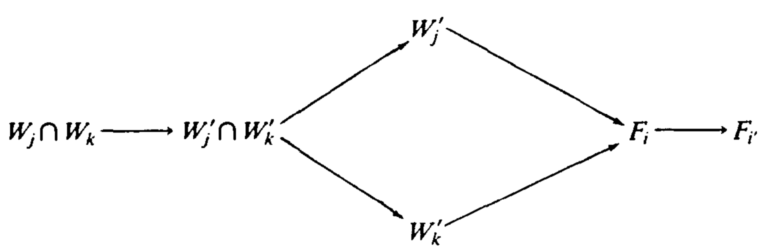
Proof. Let $(F_i)_{i \in I}$ be a filtered diagram of sheaves with colimit V. If σF_i denotes the support of F_i, i.e. the image of the unique map $F_i \to 1$, then the σF_i form a directed family of open sets with join V, and so we can find $i \in I$ such that $U \subseteq \sigma F_i$ – in fact, using the subdivisibility of $\ll$, we can even achieve $U \ll \sigma F_i$. Now σF_i is covered by the open sets over which F_i admits a section, and hence by the sets

$$\{ W \mid (\exists W' \gg W)(F_i \text{ admits a section over } W') \},$$

so we can find a finite subfamily $\{W_1, \ldots, W_n\}$ of these sets which covers U. For each $j \leq n$, let s_j be a section of F_i over an open set $W_j' \gg W_j$. For $j \neq k$, the two maps



need not be equal, but they become equal when composed with the canonical map $F_i \to \varinjlim_i F_i \cong V$. So if $E_{i'} \to W_j' \cap W_k'$ denotes the equalizer of their composites with a map $F_i \to F_{i'}$ of the filtered diagram, then the $E_{i'}$ form a directed family of subsets of $W_j' \cap W_k'$ whose join is the whole of $W_j' \cap W_k'$. But by stability we have $W_j \cap W_k \ll W_j' \cap W_k'$, so there exists i' with $W_j \cap W_k \subseteq E_{i'}$ – i.e. such that the two composites



are equal. Repeating this argument for each pair (j,k), we arrive at an $F_{i''}$ and a family of sections $W_j \to F_{i''}$ which are pairwise compatible – so that they can be patched together to obtain a section of $F_{i''}$ over $\bigcup_{j=1}^{n} W_j$. But the W_j cover U, so we have a section of $F_{i''}$ over U. □

Example 5.3. If the way-below relation is not stable, then the conclusion of Proposition 5.2 is false. Let X be the space obtained by identifying two disjoint copies of the unit interval $[0,1]$ along the open subspace $[0,1)$ (cf. [34], Example 73); then it is easily seen that X is compact and locally compact, but not Hausdorff. For $0<t<1$, we may similarly define X_t to be the space obtained by identifying two copies of $[0,1]$ along $[0,t)$; then for $t<t'$ there is an obvious local homeomorphism $X_t \to X_{t'}$, and thus we can regard the X_t as a directed diagram of sheaves over $X_1 = X$. Moreover the colimit $\varinjlim_{t<1} X_t$ is homeomorphic to X; but none of the X_t admits a section over X. So we have $X \ll X$ (by compactness) but not $X \lll X$.

It may be shown that if U_1 and U_2 denote the two copies of $[0,1]$ embedded in the space X of Example 5.3, then we do have $U_1 \lll X$ and $U_2 \lll X$. But $U_1 \cup U_2 = X$; thus the relation $\lll$, unlike $\ll$, is not in general stable under finite joins.

We shall say that a locale X is *metastably locally compact* if every open $V \subseteq X$ can be covered by open sets U satisfying $U \lll V$. Thus Proposition 5.2 tells us that stably locally compact locales are metastably locally compact.

Lemma 5.4. *Let X be a locale such that* Shv(X) *is a continuous category. Then X is metastably locally compact.*

Proof. Given V, let $L(V) = (F_i)_{i \in I}$. As in the proof of Proposition 5.2, we know that V is the join of the supports σF_i, $i \in I$; and each σF_i can be covered by open sets U over which F_i admits a section. But if F_i admits a section over U then $U \lll V$. □

Our main objective in this section is to show that the converse of Lemma 5.4 is true. Before embarking on this, however, we give our promised example of a locally compact space X such that Shv(X) is not exponentiable. It should be thought of as an 'iterated' version of the space of Example 5.3.

Example 5.5. Let X be the quotient of the space $[0,1] \times 2^N$ (where 2^N denotes the Cantor space) by the equivalence relation R, where

$$(x,s)\,R\,(y,t) \Leftrightarrow x=y, \quad \text{and} \quad \begin{array}{l} \text{either } s=t \\ \text{or there exists } n \text{ such that} \\ \quad x<1-1/2^{n+1} \text{ and } s|_n = t|_n \end{array}$$

(here $s|_n$ denotes "the first n terms of the sequence s"). In terms of its canonical projection onto $[0,1]$, X has a discrete 2^n-point space as its fibre over each point of the interval $[1-1/2^n, 1-1/2^{n+1})$, and a copy of Cantor space as its fibre over 1. It is not hard to see that the quotient map $q:[0,1]\times 2^n \to X$ is an open map, and hence that X is locally compact. However, we shall show that no open set U satisfying $U \lll X$ meets the fibre over 1.

For if U is a neighbourhood of the point $q(1,s)$, then there exists n such that U also contains the points $q(1-1/2^{n+1},s)$ and $q(1-1/2^{n+1},s')$ where s' differs from s at the $(n+1)$st term but not before. Thus U contains an open subspace U' which looks like the effect of identifying two copies of an open interval $(t-\varepsilon,t+\varepsilon)$ (where $t=1-1/2^{n+1}$ and $0<\varepsilon<1/2^{n+2}$) along the subinterval $(t-\varepsilon,t)$. For $0<\varepsilon'<\varepsilon$, let $X_{\varepsilon'}$ be the space obtained from X by 'unglueing' these two intervals over $[t-\varepsilon',t)$. Then it is clear that the $X_{\varepsilon'}$, $\varepsilon'>0$, form a directed system of sheaves on X with colimit X. But none of the $X_{\varepsilon'}$ admits a section over U', let alone over U, so we do not have $U \lll X$.

Thus we have shown that X is not metastably locally compact; hence by Lemma 5.4 Shv(X) is not a continuous category, and so by Lemma 4.6 it is not exponentiable.

We now embark on proving the converse of Lemma 5.4. First we note that metastable local compactness is a local property (i.e. it holds for X iff it holds for each member of a covering family of open subspaces of X), and thus if X is metastably locally compact, so is the domain of any local homeomorphism $E \to X$. Thus we may reduce the problem of constructing the functor $L:\mathrm{Shv}(X)\to$ Ind-Shv(X) (i.e. of constructing $L(E)$ for every such E) to that of constructing the particular ind-object $L(X)$; and for this it suffices by Corollary 2.3 to construct an initial object in the category of ind-objects with colimit X.

We define a category $\mathscr{C}$ as follows: its objects are diagrams $(F\to G)$ where G is a sheaf on X such that for every filtered diagram of sheaves $(H_i)_{i\in I}$ with $\varinjlim_i H_i \cong X$, there exists a morphism $G\to H_i$ for some i, and F is a subsheaf of G satisfying $F \ll G$ in the (continuous) lattice of subsheaves of G. A morphism $f:(F_1\to G_1)\to(F_2\to G_2)$ in $\mathscr{C}$ is a sheaf morphism $f:F_1\to F_2$ for which there exists some $g:G_1\to G_2$ making the diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{f} & F_2 \\ \downarrow & & \downarrow \\ G_1 & \xrightarrow{g} & G_2 \end{array}$$

commute. As it stands, the category $\mathscr{C}$ is obviously not small; but we can cut down to an essentially small full subcategory $\mathscr{C}_0$ by imposing the additional restriction on objects that G can be covered by a finite number of sections, i.e. there exists an epimorphism

$$\sum_{i=1}^{n} U_i \to G$$

where the U_i are subobjects of 1 in Shv(X).

Lemma 5.6. *The category $\mathscr{C}_0$ is filtered.*

Proof. It is clearly nonempty. Given two objects $(F_1 \to G_1)$ and $(F_2 \to G_2)$, we may form the coproduct $(F_1 + F_2 \to G_1 + G_2)$. Then in the lattice of subobjects of $G_1 + G_2$ we have $F_1 \ll G_1 \leq G_1 + G_2$ and $F_2 \ll G_2 \leq G_1 + G_2$, whence $F_1 + F_2 \ll G_1 + G_2$; and similarly if we have a filtered diagram $(H_i)_{i \in I}$ with $\varinjlim_i H_i \cong X$, then morphisms $G_1 \to H_i$ and $G_2 \to H_{i'}$ may be combined to produce $G_1 + G_2 \to H_{i''}$ for some $i'' \in I$. Also, $G_1 + G_2$ may be covered by the disjoint union of the finite sets of sections which cover G_1 and G_2; and the coproduct inclusions $F_i \to F_1 + F_2$ clearly define morphisms $(F_i \to G_i) \to (F_1 + F_2 \to G_1 + G_2)$ in $\mathscr{C}_0$.

Now suppose we are given a parallel pair of morphisms $f_1, f_2 : (F_1 \to G_1) \to (F_2 \to G_2)$ in $\mathscr{C}_0$. Form the diagram

$$\begin{array}{ccccc} F_1 & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & F_2 & \xrightarrow{f_3} & F_3 \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \overset{g_1}{\underset{g_1}{\rightrightarrows}} & G_2 & \xrightarrow{g_3} & G_3 \end{array}$$

where the top row is a coequalizer and the right-hand square is a pushout. By a well-known property of pushouts in a topos [20, Corollary 1.28], the morphism $F_3 \to G_3$ is mono, and the square is also a pullback. In addition, g_3 is epi, so the pullback functor $g_3^* : \mathrm{Shv}(X)/G_3 \to \mathrm{Shv}(X)/G_2$ is faithful. Now if we are given a directed family $(U_i)_{i \in I}$ of subsheaves of G_3 with join G_3, then we have $\bigvee_{i \in I} g_3^*(U_i) = G_2$, and so there exists $i \in I$ with $g_3^* F_3 \cong F_2 \leq g_3^* U_i$. But then faithfulness of g_3^* tells us that $F_3 \leq U_i$; so we have shown that $F_3 \ll G_3$ in the lattice of subobjects of G_3.

Next, consider a filtered diagram of sheaves $(H_i)_{i \in I}$ with colimit X. By assumption, there exists a map $h : G_2 \to H_i$ for some i. The composites $hg_1, hg_2 : G_1 \to H_i$ need not be equal, but they are coequalized by $H_i \to \varinjlim_i H_i$, so the equalizers of their composites with maps $H_i \to H_{i'}$ of the filtered diagram form a directed family of subobjects of G_1 with join G_1. In particular, we can find $H_i \to H_{i'}$ for which the equalizer contains F_1, i.e. such that the composites

$$F_1 \overset{f_1}{\underset{f_2}{\rightrightarrows}} F_2 \longrightarrow G_2 \xrightarrow{h} H_i \longrightarrow H_{i'}$$

are equal. But then we can factor $F_2 \to H_{i'}$ through the coequalizer f_3, i.e. we obtain a commutative diagram

$$\begin{array}{ccc} F_2 & \xrightarrow{f_3} & F_3 \\ \downarrow & & \downarrow \\ G_2 \xrightarrow{h} H_i & \longrightarrow & H_{i'} \end{array}$$

and hence a factorization $G_3 \to H_{i'}$ through the pushout. So $(F_3 \to G_3)$ is an object of $\mathscr{C}$ – in fact of $\mathscr{C}_0$, since G_3 is a quotient of G_2 – and $f_3 : (F_2 \to G_2) \to (F_3 \to G_3)$ is a morphism of $\mathscr{C}_0$ coequalizing f_1 and f_2. □

We have an obvious forgetful functor $T: \mathscr{C}_0 \to \mathrm{Shv}(X)$ sending $(F \to G)$ to F; in view of Lemma 5.6, we may regard this as an object of Ind-Shv(X).

Lemma 5.7. *If X is metastably locally compact, then the colimit of the ind-object T defined above is isomorphic to X.*

Proof. X may be covered by open sets U for which there exists a V with $U \ll V \lll X$; but for every such U the inclusion $(U \to V)$ is an object of $\mathscr{C}_0$, and hence the colimit of T must have global support. To show that the colimit is a subobject of 1, it suffices to show that for every V in some basis of open sets, each pair of maps $V \rightrightarrows T(F \to G)$ is coequalized by the map $T(F \to G) \to \varinjlim T$. But if we choose V so that $V \lll X$, then for each $U \ll V$ the composites $U \to V \rightrightarrows F$ may be regarded as morphisms $(U \to V) \rightrightarrows (F \to G)$ in $\mathscr{C}_0$, so by Lemma 5.6 there is a map $(F \to G) \to (F' \to G')$ coequalizing them. But V is covered by such open sets U, so the equalizer of $V \rightrightarrows T(F \to G) \to \varinjlim T$ is the whole of V. □

Lemma 5.8. *Under the hypotheses of Lemma 5.7, the ind-object T is initial among ind-objects with colimit X.*

Proof. Let $(H_i)_{i \in I}$ be an arbitrary ind-object with colimit X, and let $(F \to G)$ be an object of $\mathscr{C}_0$. By definition, there exists a map $h : G \to H_i$ for some $i \in I$; there may be many such maps, but if h_1 and h_2 are two such, then there exists a morphism $H_i \to H_{i'}$ of the filtered diagram such that the equalizer of

$$G \underset{h_2}{\overset{h_1}{\rightrightarrows}} H_i \longrightarrow H_{i'}$$

contains F. Hence the restrictions to F of h_1 and h_2 are equivalent as maps into the filtered diagram; that is,there is a *unique* equivalence class of maps $F \to H_i$ which includes the restriction to F of some morphism defined on G. Moreover, it is clear that if we take this distinguished equivalence class for every object $(F \to G)$ of $\mathscr{C}_0$, we obtain a morphism of ind-objects $T \to (H_i)_{i \in I}$.

We must show that this is the unique such morphism. But if $(F \to G)$ is any object of $\mathscr{C}_0$, we can find a subobject F' of G with $F \ll F' \ll G$, and then we have a morphism $(F \to G) \to (F' \to G)$ in $\mathscr{C}_0$. So the equivalence class of maps $F \to H_i$ assigned to the object $(F \to G)$ by a morphism of ind-objects $T \to (H_i)_{i \in I}$ must contain some morphism which extends to F'; and by the argument given above, this is sufficient to determine it uniquely. □

Putting together the results of the last four lemmas, we have proved:

Theorem 5.9. *Let X be a locale. The topos* Shv(X) *is a continuous category iff X is metastably locally compact.*

Proof. One direction is Lemma 5.4. In the converse direction, Lemma 5.8 tells us that there is an initial ind-object with colimit X, and Corollary 2.3 says that this ind-object must be $L(X)$; and as explained after Example 5.5, the stability of the hypotheses under localization then ensures that $L(F)$ exists for every sheaf F on X. □

Corollary 5.10. *If X is a stably locally compact space (for example a locally compact Hausdorff space), then the topos* Shv(X) *is exponentiable in* $\mathfrak{BTop}/\mathcal{S}$. □

It remains to give a more locale-theoretic (i.e. less sheaf-theoretic) description of the relation $\ll$. Now in Chapter 2 of [5], K.R. Edwards investigated the condition that the global section functor Shv(X)$\to\mathcal{S}$ preserves filtered colimits (which in our terminology is simply the assertion that $X \ll X$); by adapting her characterization to our more general context, we obtain

Proposition 5.11. *Let U and V be open sets in a locally compact locale X. Then $U \ll V$ iff the following condition holds:*

(♦) *Given any open cover $(V_\alpha)_{\alpha\in A}$ of V, there exists a finite $B\subseteq A$ and open sets $U_\alpha \subseteq V_\alpha \cap U$ $(\alpha\in B)$, $W_{\alpha\beta}\subseteq U_\alpha\cap U_\beta$ $(\alpha,\beta\in B)$ such that $W_{\alpha\beta} \leqslant V_\alpha \cap V_\beta$ for each (α,β) and the canonical diagram*

$$\sum_{\alpha,\beta\in B} W_{\alpha\beta} \rightrightarrows \sum_{\alpha\in B} U_\alpha \to U$$

is a coequalizer in Shv(X). [*Informally, U may be constructed by patching together the members of a finite refinement of $(V_\alpha)_{\alpha\in A}$ over sets which are way below the pairwise intersections of the V_α.*]

Proof. First we show the necessity of (♦). Given an open cover $(V_\alpha)_{\alpha\in A}$ of V, consider the set of all sheaves F which can be formed as coequalizers

$$\sum_{\alpha,\beta\in B} T_{\alpha\beta} \rightrightarrows \sum_{\alpha\in B} V_\alpha \to F$$

where B runs over all finite subsets of A and $T_{\alpha\beta} \leqslant V_\alpha \cap V_\beta$ for each (α,β). We can make these into the vertices of a directed diagram, in which there exists a morphism $F\to F'$ iff (in the obvious notation) we have $B\subseteq B'$ and $T_{\alpha\beta}\subseteq T'_{\alpha\beta}$ for each $(\alpha,\beta)\in B\times B$; and since each $V_a\cap V_b$ is covered by open sets which are way below it, it is easily verified that the colimit of this diagram is V. So if $U \ll V$, there exists a morphism $h: U\to F$ for some such F; then we merely define $U_\alpha = h^*(V_\alpha)$ and $W_{\alpha\beta}=h^*(T_{\alpha\beta})$ to obtain the desired properties, since coproducts and coequalizers are preserved under pullback in Shv(X).

The proof of sufficiency of (♦) is similar to the argument used in proving

Proposition 5.2. Let $(H_i)_{i\in I}$ be an arbitrary filtered diagram of sheaves with colimit V; then we can cover V by open sets V_α over each of which some H_i admits a section s_α. Choose open sets U_α and $W_{\alpha\beta}$ as in (♦); since we now have a finite set B of indices to deal with. We may assume that the sections s_α $(\alpha\in B)$ all take values in the same H_i. By an argument we have used several times before, we may now find for each pair (α,β) a morphism $H_i \to H_{i'}$ of the filtered diagram such that the equalizer of

$$\begin{array}{ccccc} & & V_\alpha & & \\ & \nearrow & & \searrow^{s_\alpha} & \\ V_\alpha \cap V_\beta & & & & H_i \longrightarrow H_{i'} \\ & \searrow & & \nearrow_{s_\beta} & \\ & & V_\beta & & \end{array}$$

contains $W_{\alpha\beta}$; choosing $H_i \to H_{i'}$ so that this happens for all pairs (α,β) simultaneously, we deduce that the composite

$$\sum_{\alpha\in B} U_\alpha \to \sum_{\alpha\in B} V_\alpha \to H_i \to H_{i'}$$

factors through the coequalizer $\sum_{\alpha\in B} U_\alpha \to U$ of $\sum W_{\alpha\beta} \rightrightarrows \sum U_\alpha$. So $H_{i'}$ admits a section over U, and hence $U \lll V$. □

As stated, the condition (♦) of Proposition 5.11 does still make reference to sheaves on X. The trouble is that we cannot in general require the diagram

$$\sum_{\alpha,\beta} W_{\alpha\beta} \rightrightarrows \sum_{\alpha} U_\alpha \to U$$

to be a kernel-pair as well as a coequalizer; that is, we cannot demand that $W_{\alpha\beta} = U_\alpha \cap U_\beta$ for all (α,β). It is possible to make $\sum W_{\alpha\beta}$ reflexive and symmetric as a relation on $\sum U_\alpha$; but when we try to take its transitive closure, we face the problem that (in the absence of stability) the relations $W_{\alpha\beta} \lll V_\alpha \cap V_\beta$ and $W_{\beta\gamma} \lll V_\beta \cap V_\gamma$ do not imply $W_{\alpha\beta} \cap W_{\beta\gamma} \lll V_\alpha \cap V_\gamma$. If we wish to remove all mention of sheaves from the condition (♦), we can do so by making explicit what it mans for the transitive closure of $\sum W_{\alpha\beta}$ to be the kernel-pair of $\sum U_\alpha \to U$; i.e. we may replace the last clause of (♦) by the condition:

For each pair (α,β), $U_\alpha \cap U_\beta$ is covered by the sets

$$W_{\alpha,\beta_1} \cap W_{\gamma_1,\gamma_2} \cap \cdots \cap W_{\gamma_{n-1},\gamma_n} \cap W_{\gamma_n,\beta}$$

where $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ $(n \ge 0)$ runs over all finite strings of members of the index set B.

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